Tricky Problems

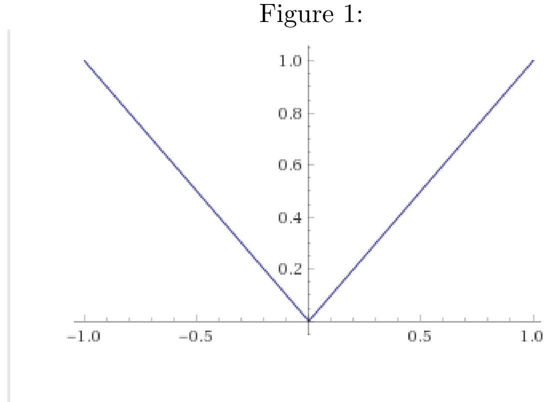
7.4: Find each of the following for the density function

$$f(x) = \begin{cases} |x| & \text{for } -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. **The cumulative distribution function F.**

They key to this problem is to note that the absolute value function is discontious at a point over this domain, specifically at 0. To see this note what the graph of this function looks like:

Figure 1:



Thus we will have to solve this piecewise, integrating over each interval and then combining that information to form our CDF. Beginning with the first interval for which we have positive probability:

$$F(x) = \int_{-1}^{x} |t|dt = \int_{-1}^{x} -tdt$$

Important to note is that over this domain the function we are integrating over is -t. Taking the antiderivative and evaluating the integral:

$$\int_{-1}^{x} -tdt = -\frac{1}{2}t^2 \Big|_{-1}^{x} = \frac{1}{2} - \frac{1}{2}x^2$$

Now, noting the trivial cases of $x < -1$ and $x \geq 1$, we get most of our CDF:

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2} - \frac{1}{2}x^2 & \text{for } -1 \leq x < 0 \\ & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Turning to the interval between 0 and 1 we get:

$$\int_{0}^{x} tdt = \frac{1}{2}t^2 \Big|_{0}^{x} = \frac{1}{2}x^2$$

But note, this is not the CDF function for this range! Recall that when we were calculating the CDF for discrete random variables from the PMF how the CDF value would be the sum of the probabilities for each of the previous values. Here we must do the same thing, which requires us to integrate the PDF over its entire range. As such, the CDF function over this range is equal to:

$$\int_{-1}^{0} -tdt + \int_{0}^{x} tdt = \frac{1}{2} + \frac{1}{2}x^2$$

Thus we get our final CDF of:

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2} - \frac{1}{2}x^2 & \text{for } -1 \leq x < 0 \\ \frac{1}{2} + \frac{1}{2}x^2 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

7.6: Find each of the following for the density function

$$f(x) = \begin{cases} e^{-x} & \text{for } 0 \leq x \leq \infty \\ 0 & \text{for } x < 0 \end{cases}$$

a. **The cumulative distribution function F.**

Here the problem is tricky not because we are dealing with any piecewise functions, but because of the strange properties of e^{-x} when we are taking derivatives or integrals. You may recall from your prior calculus experience that for some random variable x and some constant a:

$$\frac{\partial}{\partial x} e^{ax} = ae^{ax}$$

Thus we know that:

$$\frac{\partial}{\partial x} e^{-x} = -e^{-x}$$

And consequently (ignoring the constant of integration):

$$\int e^{-x} dx = -e^{-x}$$

With the e^{-x} function, both the derivative and antiderivative flip the sign of the function. Using this information we can begin to calculate our CDF. Like always, we begin by integrating the PDF from the lower bound to x:

$$\int_0^x e^{-x} dx = -e^{-x}\Big|_0^x = -e^x - (-e^0)$$

Now, recall that anything to the 0^{th} power is equal to 1. Thus with the double negative we get:

$$\int_0^x e^{-x} dx = 1 - e^{-x}$$

And consequently our CDF:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-x} & \text{for } 0 \leq x < \infty \end{cases}$$