Econ 204A: Section 3

Ryan Sherrard

University of California, Santa Barbara

18 October 2016

Notes on Problem Set 2

- Total Derivative Review

$$sf(k^*) = (\delta + n + g)k^*$$

$$f(k^*)ds + sf'(k^*)dk^* = (\delta + n + g)dk^* + (d\delta + dn + dg)k^*$$

- Always look at the solutions.
- Thomas fix your header.

Motivation

- The Solow model assumes TFP growth is exogenous but we know growth comes from *within* economies

- **New growth** attempts to endogenize TFP growth in various ways
  - Focus on Romer model

- Need to kill diminishing marginal returns to obtain steady-state growth
  - **Solow**: $Y/L = A^{1-\alpha}(K/L)^{\alpha} \Rightarrow$ growing scale factor
  - **AK**: $Y = A \cdot K \Rightarrow$ growing by assumption
  - **Lucas**: $\dot{h} = (1-u)h$ & $Y = K^{\alpha}(hu \cdot L)^{1-\alpha} \Rightarrow$ linear $\dot{h}$ skirts the issue
  - **Romer**: $\dot{A} = \delta \cdot A^{\phi} \cdot L_A^{\lambda} \Rightarrow$ new products drive growth

Romer-Jones: Set-up

- Interpreting A as patentable ideas is key mechanism:
  - New innovations $\Rightarrow$ start over with high returns
  - Many new products $\Rightarrow$ "endogenous" exponential growth

- Romer-Jones model incorporates several key assumptions
  1) Competitive final goods sector aggregates discovered intermediate goods

$$Y = (L_y)^{1-\alpha} \cdot \int_0^{A_t} x_{it}^{\alpha} di$$

  2) Research sector discovers new ideas and receives patents

$$\dot{A} = \delta \cdot A^{\phi} \cdot L_A^{\lambda}$$

  3) Monopolistically competitive intermediate sector transforms capital into intermediate goods

$$x_i = \frac{K}{A}$$

Romer-Jones: Endogenous Growth

You can easily get back to the Solow model:

$$x_i = \frac{K}{A} \qquad \Rightarrow \qquad Y = L_y^{1-\alpha} \cdot \int_0^{A_t} (K/A)^{\alpha} di$$

$$\Leftrightarrow Y = L_y^{1-\alpha} (K/A)^{\alpha} \int_0^{A_t} di$$

$$\Leftrightarrow Y = (AL_y)^{1-\alpha} \cdot K^{\alpha}$$

- We know Y/L grows due to A, which is now endogenous

$$\dot{A} = \delta \cdot A^{\phi} \cdot L_A^{\lambda}$$

Romer-Jones: Research Sector

- Assume that research is costly in that it requires labor, $L_A = L - L_y$
- Discovery of ideas is subject to two externalities:
  1. **Congestion effect**: possibility of simultaneous innovation
  2. **Spillovers**: stock of ideas help/hamper new discovery

$$\dot{A} = \delta \cdot A^{\phi} \cdot L_A^{\lambda}$$

- New ideas beget new ideas iff $\phi > 0$ but are hard to find iff $\phi < 0$
- Simultaneous innovation and/or wasted effor iff $\lambda < 1$ and perfect coordination iff $\lambda = 1$

Romer-Jones: Steady State Growth

- We know that along the balanced growth path, all variables grow by a constant rate

$$\frac{\dot{A}}{A} = g_A^* \quad \Leftrightarrow \quad \delta \cdot \frac{L_A^{\lambda}}{A^{1-\phi}} = const.$$

$$\Rightarrow \frac{L_A^{\lambda}}{A^{1-\phi}} = const.$$

$$\Rightarrow \lambda \frac{\dot{L}_A}{L_A} = (1-\phi)\frac{\dot{A}}{A}$$

$$\Leftrightarrow \lambda n = (1-\phi) g_A^*$$

$$\Leftrightarrow g_A^* = \frac{n\lambda}{1-\phi}$$

Romer-Jones: Steady States

We can derive the steady states in efficiency units as in Solow

$$k = \frac{K}{AL}$$

$$\Rightarrow \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n - g_A$$

$$\Leftrightarrow \frac{\dot{k}}{k} = \frac{sK^{\alpha}(A(1-s_R)L)^{1-\alpha} - \delta K}{K} - g_A - n$$

$$\Leftrightarrow \frac{\dot{k}}{k} = s((1-s_R)/k)^{1-\alpha} - g_A - n - \delta$$

$$\Leftrightarrow \dot{k} = s(1-s_R)^{1-\alpha}k^{\alpha} - (g_A + n + \delta)k \qquad \Longrightarrow \qquad k^* = (1-s_R)\left(\frac{s}{\delta + n + g_A^*}\right)^{\frac{1}{1-\alpha}}$$

Romer-Jones: Comparative Statics

- We can perform comparative statics just as in the Solow model
- Ex: How does the steady state in efficiency units change when the congestion parameter decreases?

$$g_A^* = \frac{n\lambda}{1-\phi} \quad \Longrightarrow \quad \Delta g_A^* < 0$$

- Recall the steady-state capital in efficiency units

$$k^* = (1-s_R)\left(\frac{s}{\delta + n + g_A^*}\right)^{\frac{1}{1-\alpha}} \quad \Longrightarrow \quad \frac{dk^*}{dg_A^*} < 0$$

- Thus, the worse the congestion effect the higher the steady state *in efficiency units*

Introduction to the Ramsey model

- This model has a lot of "names" that you may have heard before
  - Optimal Control (though, really, it's only an application of control theory)
  - Ramsey-Cass-Koopmans (RCK) model
  - Neoclassical Growth model
- Interesting notes: **Ramsey** (1928), Cass (1965), Koopmans (1965)
  - was a philosopher / mathematician / economist
  - died at the age of 26 from jaundice after a liver operation
  - genius
- This model will improve on the Solow model in an important way: we will now think seriously about agents making choices (microfoundations)

- Here, we will have a utility function specified and agents will choose between consumption and savings
- We can formulate this "problem" from the perspective of a household (HH) or a social planner (SP)
- The HH problem will be decentralized, where markets will guide the allocation of resources (because HH will be identical, we can think of a representative consumer)
- The SP problem will be centralized, where one (benevolent) agent makes all allocative decisions
- Note that, in the absence of market frictions, the allocations in the representative HH problem will be the same as in the SP problem

Things we'll encounter

- We will see a new way of setting up / solving a problem: the Hamiltonian and the maximum principle
- Just like the Solow model, the Ramsey model has a nice graphical representation: the phase diagram
- In the process of solving the model, we will need to make use of some simple differential equation results
- A note on notation: Y (agg. output), I (agg. investment), L (population), H (number of households), C (per-capita consumption!)

The Social Planner's Problem

First let's think of a SP's problem; all HHs will be identical, so we can also think of this as a representative consumer who has access to all productive capabilities in the economy.

$$\max U = \int_0^\infty \left[e^{-\rho t} u(C) L \right] dt \qquad \text{s.t.} \qquad \dot{K} = F(K, AL) - CL - \delta K$$

where ρ is the rate of time preference. Because we will eventually want to think about a steady state, it is customary to transform the above problem into effective units just like in Solow: $x = X/AL$ for most variables and $c = C/A$ for consumption. For the constraint we'll have

$$\dot{k} = f(k) - c - (\delta + n + g)k.$$

The Houshold's Problem

Next, consider the problem of a HH. They wish to maximize their HH utility subject to a budget constraint.

$$\max U = \int_0^\infty \left[e^{-\rho t} u(C) \frac{L}{H} \right] dt \qquad \text{s.t.} \qquad \dot{a} = ra + W\frac{L}{H} - C\frac{L}{H}$$

Notice the similarities in the setup. Solving this problem will be fairly similar to the SP problem, with some minor key differences that we'll get to (like how to determine prices, for instance). For now, let's put this problem aside and think just about the SP problem (which extends from the Solow model more directly).

Solving these Problems

So how do we continue? We *could* solve this problem using the familiar Lagrangian. To do this (outlined in Romer), we have to impose some boundary conditions that will allow us to derive an intertemporal budget constraint. This is done so that the shadow value of relaxing the constraint is constant.

It turns out, there is a faster (more robust) way to do these sorts of problems. We can form the **Hamiltonian** and apply the **Maximum Principle** to get a set of differential equations that are necessary conditions for optimality. Then we can impose some **boundary conditions** to get the allocations.

So, essentially, we are "reorganizing" our typical solution technique to something that is more robust and easier to carry out.

Solution Outline

1. Define / determine the choice variables, state variables, and co-state variables and construct the Hamiltonian
   - choice: c, state: k, co-state: λ
2. Apply the Maximum Principle
   - see Bohn's notes on the connection between the Maximum Principle and the typical Lagrangian setup
3. Impose boundary conditions (initial and terminal conditions)
   - initial condition (usually given)
   - terminal conditions (transversality and/or no-ponzi)

The Hamiltonian and Maximum Principle

Hamiltonian:

$$\mathcal{H}(c, k, \lambda, t) = e^{-\rho t} u(cA)L + \lambda[f(k) - c - (\delta + n + g)k]$$

Maximum Principle:

$$(i) \quad \frac{\partial \mathcal{H}}{\partial c} = 0 \qquad (ii) \quad \frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} \qquad (iii) \quad \frac{\partial \mathcal{H}}{\partial \lambda} = \dot{k}$$

Combining the results from (i) and (ii) will yield one differential equation. Step (iii) will reproduce the constraint (another differential equation). Thus we will have (for this simple setup with only one state variable) two differential equations (in two unknowns) that are necessary for optimality.

Boundary Conditions

It turns out that we need more than just the two differential equations we obtained from the previous steps. The first thing is an **initial condition**. This is usually straightforward, as we typically assume that it's given.

Next we need some **terminal conditions**. These come partly from economic intuition and partly from some restrictions we place on types of behavior.

Thinking about assets, it is not optimal to have left over (positive) assets / capital at the end of time (or in the limit). This is called a *transversality condition*. On the other hand, it would be unreasonable to allow someone to borrow and have debt (negative assets) at the end of life. This is called a *No-Ponzi condition*.

Initial Conditions:

$$k(0) \text{ or } a(0) \text{ given}$$

Terminal Conditions:

$$k(T) = 0 \text{ or } a(T) = 0 \qquad \text{(finite horizon)}$$

$$\lambda(t)k(t) \to 0 \text{ or } \lambda(t)a(t) \to 0 \text{ as } t \to \infty \qquad \text{(infinite horizon)}$$

Note that λ is the shadow price of the constraint. That is, it tells us the utility equivalent value per unit of the constraint (capital). Thus, we require that the value of holding capital at the terminal date (or in the limit) is equal to zero as per the discussion on the last slide.