Ryan Sherrard
Econ 204A
October 5, 2018

Problem Set 1

Problem 1.1. Consider the two-period consumption model: Individuals have initial assets A, earn interest r on assets, and earn wage income (w_1, w_2). They maximize utility $U = u(c_1) + \beta u(c_2)$.

(a) Assume $u(c) = \ln(c)$

i. Solve for optimal consumption and period-1 asset holdings as functions of wage income, the interest rate, and the time discount factor, β. Discuss under what conditions a marginally higher interest rate reduces consumption.

Rather than solve this problem twice (once now and once in part b) I'll instead solve this problem for any arbitrary power utility function and plug in $\gamma = 1$ later.

$$\max \frac{1}{1-\gamma} c_1^{1-\gamma} + \frac{\beta}{1-\gamma} c_2^{1-\gamma} \qquad s.t. \qquad a_1 = A + w_1 - c1$$

$$a_2 = (1+r)a_1 + w_2 - c_2$$

Using the terminal condition $a_2 = 0$, we can recursivley solve for the intertemporal budget constraint.

$$c_1 + \frac{c_2}{1+r} = A + w_1 + \frac{w_2}{1+r}$$

We can now either set-up the lagrangian or recognize that standard power utility parameters (i.e $\gamma > 0$) imply diminishing MRS.

$$MRS = \frac{p_1}{p_2} \qquad \Leftrightarrow \qquad \left(\frac{c_2}{c_1}\right)^{\gamma} \frac{1}{\beta} = 1 + r$$

$$\Leftrightarrow c_2 = \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1}{\gamma}} c_1$$

Plugging into the intertemporal B.C.

$$c_1[1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}}] = A + w_1 + \frac{w_2}{1+r}$$

$$c_1^* = [1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}]^{-1}[A + w_1 + \frac{w_2}{1+r}]$$

$$\Rightarrow c_2^* = \frac{\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}}}{1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}}[A + w_1 + \frac{w_2}{1+r}]$$

Plugging in $\gamma = 1$,

$$\boxed{c_1^* = \frac{1}{1+\beta}[A + w_1 + \frac{w_2}{1+r}]}$$

$$\boxed{c_2^* = \frac{\beta(1+r)}{1+\beta}[A + w_1 + \frac{w_2}{1+r}]}$$

Plugging in for a_1,

$$\Rightarrow \boxed{a_1 = \frac{\beta}{1+\beta}[A + w_1] - \frac{w_2}{(1+\beta)(1+r)}}$$

To determine how consumption in each period 1 and 2 changes as we increase the interest rate, simply take the derivative with respect to r.

$$\frac{\partial c_1}{\partial(1+r)} = \frac{-w_2}{(1+\beta)(1+r)^2} \leq 0 \qquad \text{(with equality if } w_2 = 0\text{)}$$
$$\frac{\partial c_2}{\partial(1+r)} = \frac{\beta}{1+\beta}[A + w_1] \qquad (< 0 \text{ iff } A < -w_1 \text{ and } > 0 \text{ iff } A > -w_1)$$

Intuitively, as $(1+r)\uparrow$, consumption in period 2 becomes relatively less expensive. Thus, the substitution effect decreases c_1 and increases c_2. If $w_2 = 0$, then income effect is positive and cancels out the substitution effect for period 1 consumption. Otherwise, the income effect is sufficiently small (possibly negative) so that c_1 decreases. For period 2 consumption, so long as individuals have suffuciently small initial debt, the income effect is sufficiently positive (though possibly still negative) such that the substitution effect dominates and c_2 increases. Conversely, so long as individuals hold sufficiently high initial debt, the income effect will be negative and dominate the substitution effect. Hence, for consumption to decrease in both periods, a sufficient condition is that households have high initial debt holdings.

ii. Show that the dependence of period-1 consumption on (w_1, w_2) can be expressed in terms of permanent income.

Recall that permanent income is defined as

$$y^p = \frac{\sum_{t=1}^{T}(1+r)^{1-t}w_t}{\sum_{t=1}^{T}(1+r)^{1-t}}$$

Let's assume that $\beta \approx \frac{1}{1+r}$. Then,

$$c_1 = \frac{1}{1+\frac{1}{1+r}}[A + w_1 + \frac{w_2}{1+r}]$$
$$\Leftrightarrow \boxed{c_1 = \frac{1}{1+\frac{1}{1+r}}A + y^p}$$

(b) Assume $u(c) = c^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$ and $\gamma \neq 1$. Do the same as in (a). In the discussion, identify which results apply for all γ, and which ones only for γ greater or less than one.

From before,

$$\boxed{c_1^* = [1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}]^{-1}[A + w_1 + \frac{w_2}{1+r}]}$$

$$\Rightarrow \boxed{c_2^* = \frac{\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}}}{1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}}[A + w_1 + \frac{w_2}{1+r}]}$$

Plugging into our a_1 equation,

$$\Rightarrow \boxed{a_1^* = \left(1 - [1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}]^{-1}\right)[A + w_1] - [1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}]^{-1}\frac{w_2}{1+r}}$$

As before, let's determine when consumption decreases as a result of a higher interest rate. Recall that the substitution effect unambiguously decreases c_1 and increases c_2. The income effect has an ambiguous effect, which depends on whether the individual is a borrow or saver. Because the *IE* must be negative in order to decrease c_2, we know that a necessary condition for $(c_1, c_2) \downarrow$ is that individuals are borrowers. Note, however, that the *IE* must be sufficiently negative to overcome the *SE* for $c_2 \downarrow$. Hence, the necessary and sufficient condition for c_2 to decrease is also sufficient condition for c_1 to decrease.

For simplicity, I'll take a monotonic transformation of c_2 first.

$$\ln c_2^* = \frac{1}{\gamma}\ln\beta + \frac{1}{\gamma}\ln(1+r) - \ln\left(1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}\right) + \ln\left(A + w_1 + \frac{w_2}{1+r}\right)$$

$$\Rightarrow \frac{\partial \ln c_2^*}{\partial(1+r)} = \frac{1}{\gamma(1+r)} - \frac{\frac{1-\gamma}{\gamma}\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-2\gamma}{\gamma}}}{1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}} - \frac{w_2(1+r)^{-2}}{A + w_1 + \frac{w_2}{1+r}}$$

A sufficient condition for this to be negative is

$$1 < \frac{(1-\gamma)\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}}{1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}} + \frac{\gamma w_2(1+r)^{-1}}{A + w_1 + \frac{w_2}{1+r}}$$

Note that if we only considered when consumption in period 1 would decrease, we see that when $\gamma > 1$ there is an ambiguous affect on c_1. When $\gamma < 1$, however, we see that c_1 is decreasing in r:

$$\frac{\partial c_1}{\partial(1+r)} = -\left(1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}\right)^{-2}\left(\beta^{\frac{1}{\gamma}}\frac{1-\gamma}{\gamma}(1+r)^{\frac{1-2\gamma}{\gamma}}\right)\left[A + w_1 + \frac{w_2}{1+r}\right]$$
$$- \left(1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}\right)^{-1}\left[\frac{w_2}{(1+r)^2}\right]$$

To show that period-1 consumption can be written in terms of permanent income, again assume that $\beta \approx 1/(1+r)$

$$c_1^* = \frac{w_1 + \frac{w_2}{1+r} + A}{1 + \frac{1}{(1+r)^{1/\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}}$$

$$\Leftrightarrow c_1^* = \frac{w_1 + \frac{w_2}{1+r} + A}{1 + \frac{1}{1+r}}$$

$$\Leftrightarrow \boxed{c_1^* = y^p + \frac{1}{1 + \frac{1}{1+r}} A}$$

Problem 1.3. Consider and individual with utility function $U = \ln(c_1) + \beta \ln(c_2) + \beta^2 \ln(c_3)$. The person has labor incomes w_1, w_2, w_3. Let $a_{t+1} = (1+r)a_t + w_t - c_t$ denote assets carried into the next period. The interest rate r is constant and assets $a_1 > 0$ are given.

(a) Specift the budget equations, derive the intertemporal B.C. and derive first-order conditions for optimality.

The problem can be formulated as follows.

$$\max \ln(c_1) + \beta \ln(c_2) + \beta^2 \ln(c_3) \qquad s.t. \qquad \begin{aligned} a_2 &= (1+r)a_1 + w_1 - c_1 \\ a_3 &= (1+r)a_2 + w_2 - c_2 \\ a_4 &= (1+r)a_3 + w_3 - c_3 \\ a_4 &= 0 \end{aligned}$$

By recursive substitution, we find the intertemporal budget constraint:

$$\boxed{c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} = (1+r)a_1 + w_1 + \frac{w_2}{(1+r)} + \frac{w_3}{(1+r)^2)}}$$

Now, we can set up the Lagrangian.

$$\mathcal{L} = \ln(c_1) + \beta \ln(c_2) + \beta^2 \ln(c_3) + \lambda \left((1+r)a_1 + w_1 + \frac{w_2}{(1+r)} + \frac{w_3}{(1+r)^2)} - c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} \right)$$

$$\Rightarrow \mathcal{L}_{c_1} = \frac{1}{c_1} = \lambda$$

$$\Rightarrow \mathcal{L}_{c_2} = \frac{\beta}{c_2} = \frac{\lambda}{1+r}$$

$$\Rightarrow \mathcal{L}_{c_3} = \frac{\beta^2}{c_3} = \frac{\lambda}{(1+r)^2}$$

Together, these imply the following optimatlity conditions.

$$\boxed{c_2 = \beta(1+r)c_1} \qquad and \qquad \boxed{c_3 = \beta^2(1+r)^2 c_1}$$

(b) Suppose that $\beta = 1/(1+r)$, $w_t = w$ is constant. Solve for period-1 consumption as a function of labor incomes and of intial assets. Define the individual's permanent income. Determine the marginal propensity to consume in period-1 from period-1 labor income; compute the MPC value for $\beta = 0.9$

Notice that from the optimality conditions, $\beta \approx 1/(1+r) \Rightarrow c_1 = c_2 = c_3$. Plugging into the budget constraint and rearranging,

$$\boxed{c_1^* = \frac{1}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} A + w}$$

$$\boxed{c_2^* = \frac{1}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} A + w}$$

$$\boxed{c_3^* = \frac{1}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} A + w}$$

Recall the definition of permanent income from problem 1.1. Substituting in $w_t = w$, we find that $\boxed{y^p = w}$. Now, let's find the marginal propensity to consume.

$$c_1^* = \frac{1}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} \left[A + w_1 + \frac{w_2}{1+r} + \frac{w_3}{(1+r)2} \right]$$

$$\Rightarrow \boxed{\frac{\partial c_1^*}{\partial w_1} = \frac{1}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}}}$$

Notice that this is less than 1. Intuitively, this makes sense due to the permanent income hypothesis. That is that individuals will consume a constant fraction of initial wealth plus their permanent income. Notice, though, that permanent income is composed of labor-income from each period. Thus, individuals should in fact be less responsive to 1 period changes in income than to changes in their permanent income. Since $\beta = 0.9 \approx 1/(1+r)$, we find that the MPC in this example is roughly 0.37.

(c) Again assuming $\beta = 1/(1+r)$ and constant w, show that assets a_t are a declining sequence and that a_t/a_{t-1} is also decreasing over time. Can you explain why this makes sense economically? Use the same economic argument to make a conjecture about the behavior of a_t/a_{t-1} in problems with more than $T = 3$ periods, notably for $T \to \infty$.

Plugging into the asset holding equations,

$$a_2 = \frac{1 + \frac{1}{1+r}}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} a_1$$
$$a_3 = \frac{1}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} a_1$$

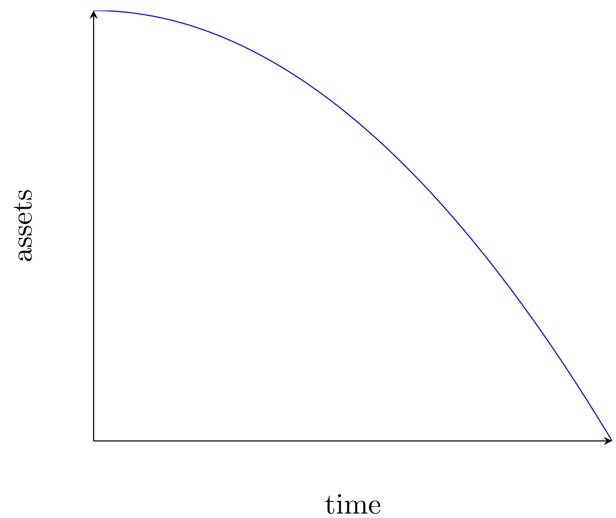
To show that assets are declining, simply show that $a_t - a_{t-1} < 0$.

$$a_2 - a_1 = -\frac{\frac{1}{(1+r)^2}}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} a_1 < 0$$
$$a_3 - a_2 = -\frac{\frac{1}{1+r}}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}} a_1 < 0$$

Thus, we see that assets are a declining series. Next, lets show that a_t/a_{t-1} is a declining series.

$$\frac{a_2}{a_1} = \frac{1 + \frac{1}{1+r}}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}}$$
$$\frac{a_3}{a_2} = \frac{1}{1 + \frac{1}{1+r}}$$
$$\Rightarrow \Delta \frac{a_t}{a_{t-1}} = -\frac{\frac{1}{1+r}}{[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}][1 + \frac{1}{1+r}]} < 0$$

Thus, a_t/a_{t-1} is a declining series. This makes sense because of the terminal condition. In other words, assets at the end of time have no value; thus, holding assets becomes less and less valuable each period. As time goes on, they begin saving less and less as result. In other words, the rate at which households decrease their assets will also increase. For a finite time intertemporal choice problem such as this one, you can imagine an asset curve that looks something like this:



For infinite time horizon problems, you can imagine the value of assets declines to 0 asymptotically.