Ryan Sherrard
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Problem Set 2

Romer Problem 1.3 Describe how, if at all, each of the following developments affects the break-even and actual investment lines in our basic diagram for the Solow model:

(a) The rate of depreciation falls.

If the depreciation rate drops, the slope of the break even investment curve falls. As a result, the break even investment line rotates clockwise. The actual investment curve, however, remains unchanged. Using our Solow graph, we know that this change should result in higher steady state capital in effeciency units. Now, lets prove it:

$$\dot{k} = sf(k) - (\delta + n + g)k$$

$$\Rightarrow sf(k^*) = (\delta + n + g)k^* \qquad \text{(Impose steady state)}$$

$$\Rightarrow sf'(k^*)\frac{dk^*}{d\delta} = k^* + (\delta + n + g)\frac{dk^*}{d\delta} \qquad \text{(Differentiate)}$$

$$\Leftrightarrow \frac{dk^*}{d\delta} = \frac{k^*}{sf'(k^*) - (\delta + n + g)} < 0 \qquad (sf'(k^*) < (\delta + n + g))$$

(b) The rate of technological progress rises.

Graphically, the break even investment line rotates counterclockwise, while the actual investment line stays is unchanged. Hence, we know that k^* decreases as capital per efficiency unit is becoming "diluted" by faster technological growth. Now, let's prove it:

$$\dot{k} = sf(k) - (\delta + n + g)k$$

$$\Rightarrow sf(k^*) = (\delta + n + g)k^* \qquad \text{(Impose steady state)}$$

$$\Rightarrow sf'(k^*)\frac{dk^*}{dg} = k^* + (\delta + n + g)\frac{dk^*}{dg} \qquad \text{(Differentiate)}$$

$$\Leftrightarrow \frac{dk^*}{dg} = \frac{k^*}{sf'(k^*) - (\delta + n + g)} < 0 \qquad (sf'(k^*) < (\delta + n + g))$$

(c) The production function is Cobb-Douglas, $f(k) = k^\alpha$, and capital's share, α, rises.

This question is slightly trickier. Because changing capital share changes the curvature of the actual investment line, there is no simple rotation or shift in either curve. Instead, the new actual investment line will be above the old curve for low capital, but below the old curve for high capital. The break even investment line does not change, however. Clearly, our graphical analysis does not give as an unambiguous answer in this case. Let's find out what happens:

$$\dot{k} = sf(k) - (\delta + n + g)k$$

$$\Rightarrow sk^{*\alpha} = (\delta + n + g)k^* \qquad \text{(Impose steady state)}$$

$$\Rightarrow s\alpha k^{*\alpha-1}dk^* + sk^{*\alpha}\ln(k^*)d\alpha = (\delta + n + g)dk^* \qquad \text{(Total derivative)}$$

$$\Leftrightarrow \frac{dk^*}{d\alpha} = \frac{sk^{*\alpha}\ln(k^*)}{(\delta + n + g) - s\alpha k^{*\alpha-1}} \qquad \text{(Rearrange)}$$

Note that in finding the total derivative the rule $\frac{d}{dx}a^x = a^x ln(a)$ might be useful. Also recognize that the denominator is always positive due to the concavity of $f(\cdot)$. The numerator on the other hand has an ambiguous sign. When $k^* > 1$, increasing capital share increases k^*. When $k^* < 1$, increasing capital share decreases k^*. The fact that the derivative has a different sign depending on where the steady state is reflects the fact that the investment line isn't simply "shifting" or "rotating."

(d) Workers exert more effort, so that output per unit of effective labor for a given value of capital per unit of effective labor is higher than before.

This change simply implies that the new actual investment curve is everywhere above the old actual investment curve (ignoring $k = 0$). For any given level of k, after this added effort is taken into account, the level of y that is produced is higher. Thus, unlike in the last part, the new output (and savings) curve, will unambiguously lie above its previous spot. Also, the break-even-investment line will remain unmoved. Thus, we will see k^* rise.

Romer Problem 1.4 Consider an economy with technological progress but without population growth that is on its balanced growth path. Now suppose there is a one-time jump in the number of workers.

(a) At the time of the jump, does output per unit of effective labor rise, fall, or stay the same? Why?

We know that a one-time change in L will not change the steady state value of k as no fundamental parameters (structure) in the economy have changed. Recall that $k = K/AL$. When L jumps up from L_0 to L_1, then k will jump down from k_0 to k_1. Next, because $y = f(k)$ where $f'(\cdot) > 0$, we know that y will also jump downwards.

(b) After the initial change (if any) in output per unit of effective labor when the new workers appear, is there any further change in output per unit of effective labor? If so, does it rise or fall? Why?

After the immediate drop in k and y, the economy will not be in the steady state. We know, however, that the economy will converge to its steady state over time. During this transition, the dynamics of the economy are governed by

$$\dot{k} = sf(k) - (\delta + n + g)k$$

Because k drops immediately after the change, $k_1 < k^*$. Hence, we know that $sf(k) > (\delta + n + g)$ during the transition. Thus, capital per efficieny unit will be increasing. Moreover, because $y = f'(k) > 0$, this implies that output per efficiency unit will also be growing over time.

(c) Once the economy has again reached a balanced growth path, is output per unit of effective labor higher, lower, or the same as it was before the new workers appeared? Why?

As mentioned in part a, the fundamental structure of the economy has not changed due to this rise in workers. In other words, we know that no fundamental parameters have change. Thus, the steady state values of k, y, and c will all be the same as before. For extra practice, see if you can draw the time evolution of K/L and Y/L as a result of this change.

Romer Problem 1.6 Consider a Solow economy that is on its balanced growth path. Assume for simplicity that there is no technological progress. Now suppose that the rate of population growth falls.

(a) What happens to the balanced-growth-path values of capital per worker, output per worker, and consumption per worker? Sketch the paths of these variables as the economy moves to its new balanced growth path.

Note here that because there is no technologial progress, the time paths of capital and output per efficiency unit is qualitatively the same as capital and output per capita. First, let's find out how our steady state values change.

$$sf(k^*) = (\delta + n)k^* \qquad \text{(Impose steady state)}$$

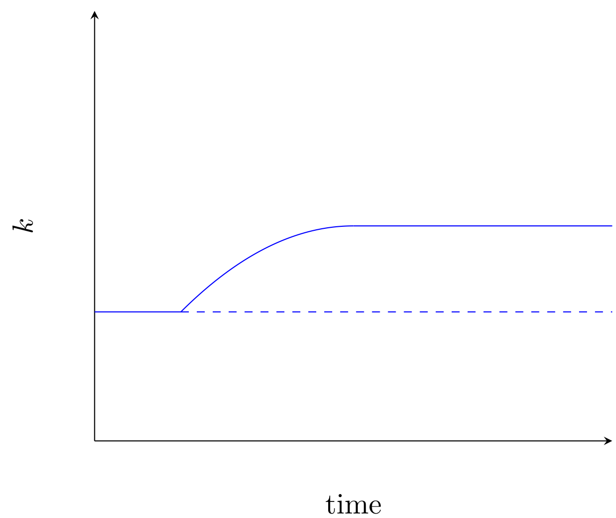
$$\Rightarrow sf'(k^*)dk^* = (\delta + n)dk^* + k^*dn \qquad \text{(Total derivative)}$$

$$\Leftrightarrow \frac{dk^*}{dn} = \frac{k^*}{sf'(k^*) - (\delta + n)} < 0 \qquad (sf'(k^*) < \delta + n)$$

Thus, as population growth increases, the steady state value of capital per worker decreases. Next, for output per worker, we know that $y^* = f(k^*)$ at the steady state. Thus

$$\frac{dy^*}{dn} = f'(k^*)\frac{dk^*}{dn} < 0,$$

Last, we know that $c^* = (1-s)f(k^*)$. The derivative w.r.t. n will be proportional to y^* with the same sign. Thus, consumption per worker will also decrease. Here, n is decreasing. As a result, k^*, y^*, and c^* all increase. The time path for capital per worker is given below. The time paths for the other variables are qualitatively identical.



(b) Describe the effect of the fall in population growth on the path of output (that is, total output, not output per worker).

Total output can be written as $Y = f(k)L = yL$. Here, I'm simply normalizing $A = 1$ since it's not growing and is thus a simple scale factor. Using the usual trick of taking logs and time derivatives, we can sketch the time path of Y.

$$ln(Y) = ln(y) + ln(L) \qquad \text{(Def.)}$$

$$\Rightarrow \frac{\dot{Y}}{Y} = \frac{\dot{y}}{y} + \frac{\dot{L}}{L} \qquad \text{(Differentiate)}$$

We know what the growth rate of labor is, but what about $\dot{y}/y$? Consider the following,

$$y = f(k) \quad \Longrightarrow \quad ln(y) = ln(f(k)) \quad \Longrightarrow \quad \frac{\dot{y}}{y} = \frac{f'(k)}{f(k)}\dot{k}$$

$$\Leftrightarrow \frac{\dot{y}}{y} = \alpha(k)\frac{\dot{k}}{k} \qquad \text{(Mulitply by } \tfrac{k}{k}\text{)}$$

Now, let's check what happens immediately before and immediately following this change. Knowing how capital per efficiency unit changes will allow us to work backwards to Y.

$$\left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = \frac{sf(k)}{k} - (\delta + n_0) \qquad \text{and} \qquad \left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} = \frac{sf(k)}{k} - (\delta + n_1)$$

$$\left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} - \left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = n_0 - n_1 > 0$$

That is, we are looking at the instant before and the instant after the change in n. Next, consider the growth in y just after the change.

$$\Delta\frac{\dot{y}}{y} = \alpha(k)\Delta\frac{\dot{k}}{k} = \alpha(k)(n_0 - n_1) > 0$$

Now, use this relationship to find the change in the growth rate of Y.

$$\Delta\frac{\dot{Y}}{Y} = \alpha(k)\Delta\frac{\dot{y}}{y} + \Delta\frac{\dot{L}}{L} = \alpha(k)(n_0 - n_1) + (n_1 - n_0)$$

$$\Leftrightarrow \Delta\frac{\dot{Y}}{Y} = (1 - \alpha(k))(n_1 - n_0) < 0$$

Hence, we know that at the time of the change, the growth rate of Y immediately changes (i.e. there's a kink). Moreover, because $\frac{\dot{y}}{y} = 0$ in the steady state, we know that $\frac{\dot{Y}}{Y} \to \frac{\dot{L}}{L}$ along the balanced growth path. Thus, the time path of $\ln(Y)$ is the following: