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Econ 204A
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Problem Set 3

Romer Problem 2.1 Suppose an economy has a production function $y_t = k_t^\alpha$ (in efficiency units), a savings rate $s > 0$, a population growth rate of n, and a depreciation rate of δ. For parts (b)-(e) assume that the economy starts in the steady state derived in (a).

(a) Suppose $\alpha = 1/3$, $s = 0.2$, $n = 0.01$, $g = 0.01$, and $\delta = 0.04$. What are the steady state values of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit?

To solve this problem, we start as we always do. In particular, write down the definition of k, take logs of both sides, and finally take the time derivative. Lastly, impose the steady-state condition ($\dot{k} = 0$).

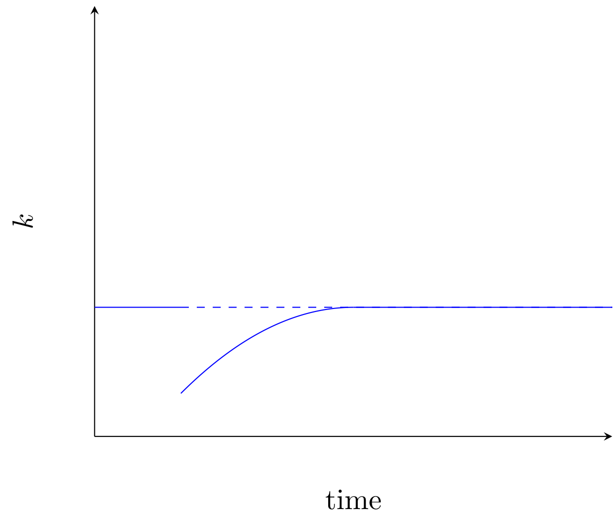
$$k^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.2}{0.04 + 0.01 + 0.01}\right)^{\frac{1}{1-.333}} = 6.0858$$

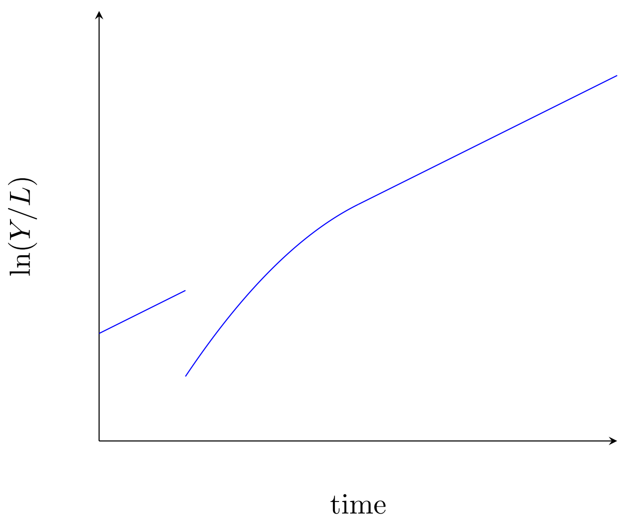
$$y^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{0.2}{0.04 + 0.01 + 0.01}\right)^{\frac{.333}{1-.333}} = 1.826$$

$$c^* = (1 - s)\left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} = (1 - 0.2)\left(\frac{0.2}{0.04 + 0.01 + 0.01}\right)^{\frac{.333}{1-.333}} = 1.461$$

(b) Suppose an earthquake destroys 10% of the caiptal stock. Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a). [To clarify: per-capita means per actual worker, not in efficiency units.]

First note the time path of k. We know at t_0 k will discretely drop. It will then increase until reaching our previous steady-state level of k. At the time of the change, we know that Y/L will also drop descretely (recall that $F_K > 0$). Because the structure of the economy has not changed, k^* remains the same. Moreover, the growth rate of per-capita variables in the long-run will not change. Lastly, we know that $k|_{t+\epsilon} < k^* \Rightarrow \frac{\dot{k}}{k} > 0$. Hence, the time paths of our per-capita variables are given by





(c) Suppose savings are increased to $s = 0.22$. What is the impact on consumption? What are the new steady state values of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

To see what the effect on k^*, y^*, and c^* is, we can plug into the steady state expressions as we did before.

$$k^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.22}{0.04 + 0.01 + 0.01}\right)^{\frac{1}{1-.333}} = 7.0211$$

$$y^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{0.22}{0.04 + 0.01 + 0.01}\right)^{\frac{.333}{1-.333}} = 1.915$$

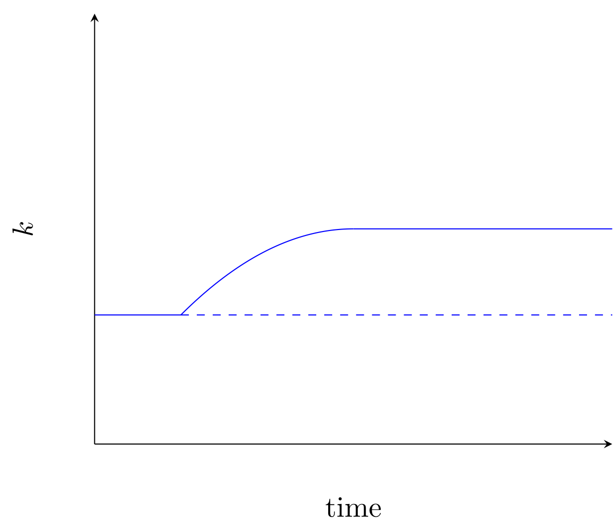
$$c^* = (1-s)\left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} = (1-0.22)\left(\frac{0.22}{0.04 + 0.01 + 0.01}\right)^{\frac{.333}{1-.333}} = 1.494$$

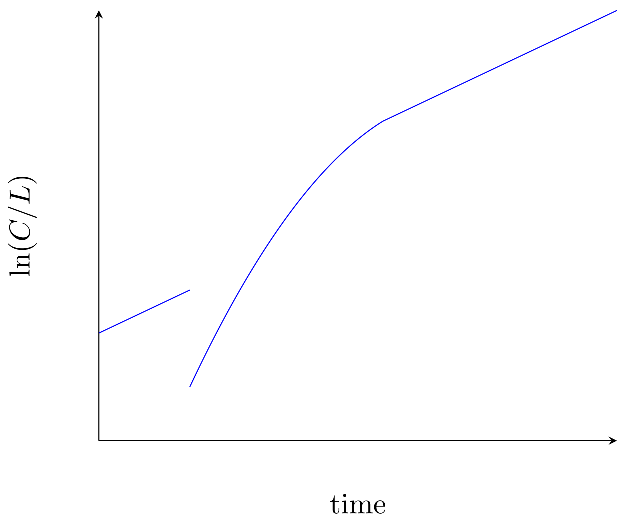
Thus, we can see that all of our steady state quantities have increased. The time paths are a little more tricky. First, recall the dynamics of capital

$$\dot{k} = sf(k) - (\delta + n + g)k$$

When s increases, $\dot{k}$ will be strictly positive right after the change. That is, the growth rate in k will go from zero to something positive. This also means that the grown rate of K/L will go from g to something strictly higher than g. In the long run we know that the growth rate of K/L will return back to g (when the steady state is reached again).

Now we can turn to consumption. Note that the change in s will not lead to an immediate level change in k nor y. Thus, we can look at consumption just after the change to see that it will have a discrete level drop because $c_{t_0+\varepsilon} = (1-0.22)y^* < (1-.2)y^*$. As we found earlier, the new steady state level of consumption will be higher than before, so in the interim we'll have c growing (and C/L growing faster than g), until the balanced growth path is reached.





(d) Suppose population growth is increased to $n = 0.02$. What are the new steady state values of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

The new steady state values will be lower than in (a).

$$k^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.2}{0.04 + 0.02 + 0.01}\right)^{\frac{1}{1-.333}} = 4.8295$$

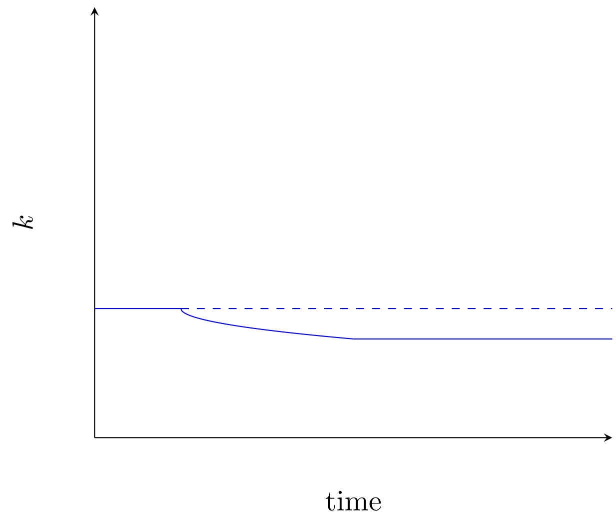
$$y^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{0.2}{0.04 + 0.02 + 0.01}\right)^{\frac{.333}{1-.333}} = 1.690$$

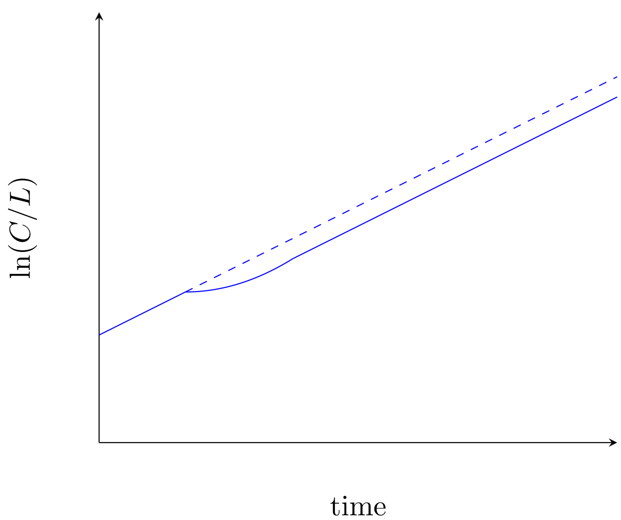
$$c^* = (1-s)\left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} = (1-0.2)\left(\frac{0.2}{0.04 + 0.02 + 0.01}\right)^{\frac{.333}{1-.333}} = 1.352$$

First note that there will not be a level drop in response to this change. Next, recall from section that we can describe the growth rate of capital as follows.

$$\frac{\dot{K/L}}{K/L} = g + \frac{\dot{k}}{k}$$

Further, recall that $\dot{k} = sk^{\alpha} - (\delta + n + g)k$. Thus, when n increases, $\dot{k}$ will be unambiguously negative. What's more, it is clear from the k^* equation that we will be at a lower steady state value of k. Thus we can see that k will decrease until reaching its lower steady state value. Turning to our per-capita variables, If you were to plug in variables, you would find that the decline in k is greater than g, thus K/L (and consequently C/L) will decline just after the change. Over time it will return to its new balanced growth path, growing at the rate g. Both time paths are given below.





(e) Suppose productivity growth is increased to $g = 0.02$. What are the new steady state values of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

Notice that the steady state values for this part will be the same as for part (d). Notwithstanding this similarity, the dynamics will look much different.

$$k^* = \left(\frac{s}{\delta+n+g}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.2}{0.04+0.01+0.02}\right)^{\frac{1}{1-.333}} = 4.8295$$

$$y^* = \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{0.2}{0.04+0.01+0.02}\right)^{\frac{.333}{1-.333}} = 1.690$$

$$c^* = (1-s)\left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} = (1-0.2)\left(\frac{0.2}{0.04+0.01+0.02}\right)^{\frac{.333}{1-.333}} = 1.352$$

Now onto the dynamics. Just as we noted before, there will not be a level drop of our variables of interest in response to a change g. Next, let's look at the dynamics of k just before, and just after, the change in g from g_1 to g_2 (where $g_1 < g_2$).

$$\left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = sk^{\alpha-1} - (n+g_1+\delta) \qquad \left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} = sk^{\alpha-1} - (n+g_2+\delta)$$

$$\left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} - \left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = g_1 - g_2 < 0$$

Now we turn to per-capita consumption. Because consumption is a fraction of output, we can just look at the dynamics of output. Recall from section that we can write the dynamics of Y/L as follows.

$$\frac{\dot{Y/L}}{Y/L} = g + \alpha\frac{\dot{k}}{k}$$

Plugging in the change in k to the above yields

$$\frac{\dot{Y/L}}{Y/L} = g_2 + \alpha[g_1 - g_2] > g_1$$

Thus the growth rate of Y/L (and C/L by extension) will be immediately greater than the initial growth rate (i.e. there will be a kink). Eventually we know that both will grow at the higher rate of g_2 in the long run. The time paths of k and C/L are given below.

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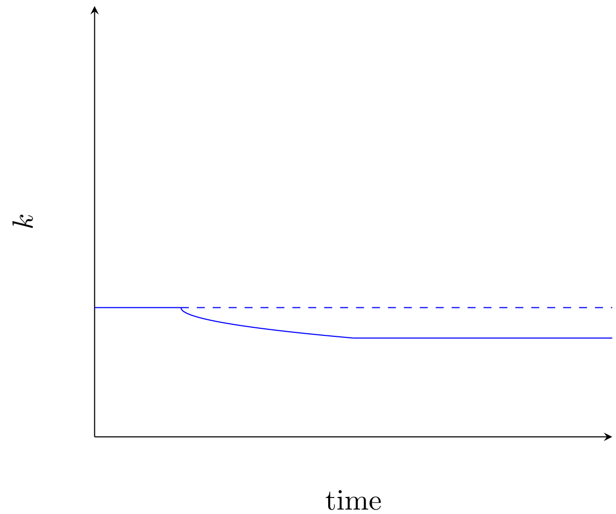
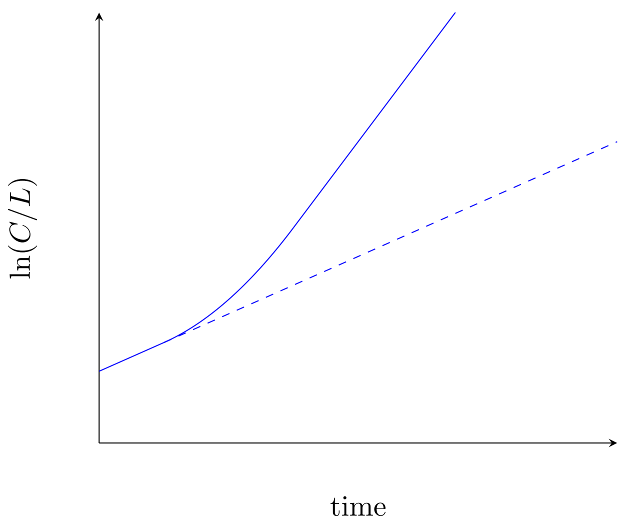




Problem 2.2 Suppose an economy has a production function $y_t = 3k_t^{0.5}$ and a savings rate of 30%, a population growth rate of 5%, and a depreciation rate of 10%. Productivity is constant.

(a) What are the steady state values of the capital-labor ratio, output per worker, and consumption per worker?

This is just as easy as it was in the last question. Skipping ahead to the "plug and chug" part.

$$k^* = \left(\frac{0.9}{0.1 + 0.05}\right)^{\frac{1}{1-0.5}} = 36$$

$$y^* = 3\left(\frac{0.9}{0.1 + 0.05}\right)^{\frac{0.5}{1-0.5}} = 18$$

$$c^* = (1 - 0.3)3\left(\frac{0.9}{0.1 + 0.05}\right)^{\frac{0.5}{1-0.5}} = 12.6$$

(b) How do the values in (a) change if the savings rate is 40%?

$$k^* = \left(\frac{1.2}{0.1 + 0.05}\right)^{\frac{1}{1-0.5}} = 64$$

$$y^* = 3\left(\frac{1.2}{0.1 + 0.05}\right)^{\frac{0.5}{1-0.5}} = 24$$

$$c^* = (1 - 0.4)3\left(\frac{1.2}{0.1 + 0.05}\right)^{\frac{0.5}{1-0.5}} = 14.4$$

(c) How do the values in (a) change with 8% population growth (still 30% savings)?

$$k^* = \left(\frac{0.9}{0.1 + 0.08}\right)^{\frac{1}{1-0.5}} = 25$$

$$y^* = 3\left(\frac{0.9}{0.1 + 0.08}\right)^{\frac{0.5}{1-0.5}} = 15$$

$$c^* = (1 - 0.3)3\left(\frac{0.9}{0.1 + 0.08}\right)^{\frac{0.5}{1-0.5}} = 10.5$$

Problem 2.3 This question is about economic growth with exogenous savings rate (s); notation is as in Romer unless noted. Assume production is Cobb-Douglas with capital share $0 < \alpha < 1$ and depreciation δ. Population L grows at rate n. Assume total factor productivity A grows at an exogenous rate g. For (b) and (c) assume that the economy at $t = 0$ is on a balanced growth path.

(a) Derive the steady state capital stock per efficiency unit of labor. Derive steady state output per efficiency unit. Derive a formula for per-capita output along the steady state growth path. [Derive means: show your work; no credit for memorized formulae.]

Starting with the definition of k, take logs and then take a time derivative.

$$k = \frac{K}{AL} \qquad \Longrightarrow \qquad ln(k) = ln(K) - ln(A) - ln(L) \qquad \Longrightarrow \qquad \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \underbrace{\frac{\dot{A}}{A}}_{g} - \underbrace{\frac{\dot{L}}{L}}_{n}$$

Next, note that capital evolves according to $\dot{K} = sY - \delta K$, where $Y = F(K, AL)$ is CRS. Plugging this into the last expression above (and then multiplying both sides by k), gives us the dynamics of capital in effective unit terms.

$$\dot{k} = sf(k) - (\delta + n + g)k \qquad \qquad \dot{k} = sk^{\alpha} - (\delta + n + g)k$$

The last statement is when production is assumed to be Cobb-Douglas. In the steady state, we know that $\dot{k} = 0$. Thus we can determine that

$$0 = sk^{*\alpha} - (\delta + n + g)k^*$$

$$sk^{*\alpha} = (\delta + n + g)k^*$$

$$k^{*1-\alpha} = \frac{s}{\delta + n + g}$$

$$k^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}.$$

$$y^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\Rightarrow \ln\left(Y/L\right) = \ln(y^*) + \ln(A)$$

$$\Leftrightarrow \ln(Y/L) = \ln(y^*) + gt + \ln(A_0)$$

(b) Suppose at time $t = 0$ that a genial discovery makes productivity A jump up by 100%. Productivity growth then continues at the original rate g.

i. Determine how the change affects the steady state y and k. Graph the time path.

A change in A will not affect the steady state values of y nor k. The doubling of A, will, however, lead to a level drop in k and, by extension, y just after the change. Eventually both of these variables will grow until the steady state level is reached again.

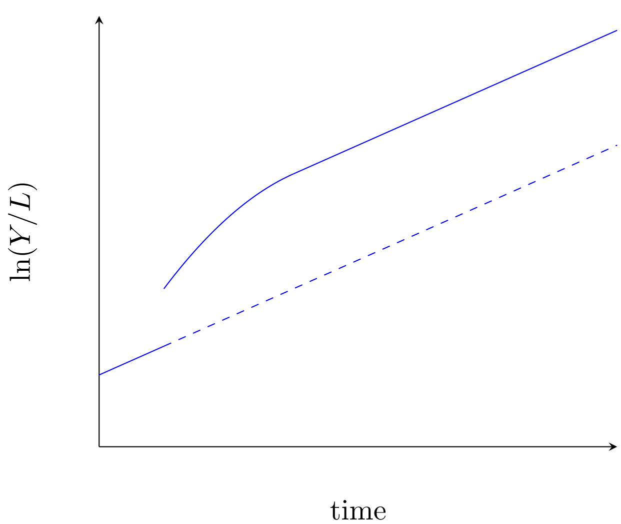
ii. Determine how the change affects Y/L and K/L. Graph the time path.

Next, recall that dynamics of k and the growth rate of K/L are given by

$$\dot{k} = sf(k) - (\delta + n + g)k$$

$$\frac{\dot{K/L}}{K/L} = g + \frac{\dot{k}}{k}.$$

The drop in k (and y) that we highlighted in the last part of the problem will mean that $\dot{k}/k > 0$. Thus, we will see that K/L will be growing just after the change. Further, we know that $K/L = Ak$, meaning that the doubling of A will cancel out the halving of k. Thus, there is no discrete change in K/L. On the other hand, we know that Y/L=$A^{1-\alpha}(Ak)^{\alpha}$. Thus, the jump in A will cause Y/L to discretely jump. Eventually we know that it will be growing at rate g. The time path $\ln(Y/L)$ is given below.



The time path of $\ln(K/L)$ is qualitatively identical, though without the discrete jump.

(c) Suppose at time $t = 0$ that the productivity growth rate increases from g_1 to g_2. There is no jump in productivity levels.

i. Determine how the change affects the steady state k. Graph the time path.

We know that the steady state level of k will decrease when g increases. We also know that this will not lead to a discrete drop. If we were to do the usual dynamics for $\dot{k}/k$, we would see that k would immediately start declining (i.e. kink), and then level off slowly until the new steady state is reached. (Note that there should not be a log around k.)

ii. Determine how the change affects Y/L and K/L. Graph the time path. Is everyone better off with higher productivity growth?

The first thing we have to determine is whether or not there is a kink. Indeed, this is very similar to some exercises we have already done (in 2.1 part (e) and in section). Skipping ahead, we know that the change in the growth rate of k in response to an increase in g can be expressed by

$$\left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} - \left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = g_1 - g_2 < 0$$

Thus, we can see how the growth rates of K/L and Y/L will be (see section slides for more steps).

$$\frac{\dot{\overline{K/L}}}{K/L} = g_2 + [g_1 - g_2] = g_1$$

$$\frac{\dot{\overline{Y/L}}}{Y/L} = g_2 + \alpha[g_1 - g_2] > g_1$$

The time path of (K/L) is given below. Note the time path of $\ln(Y/L)$ is qualitivaley similar, though with an upward kink at the time of the change.

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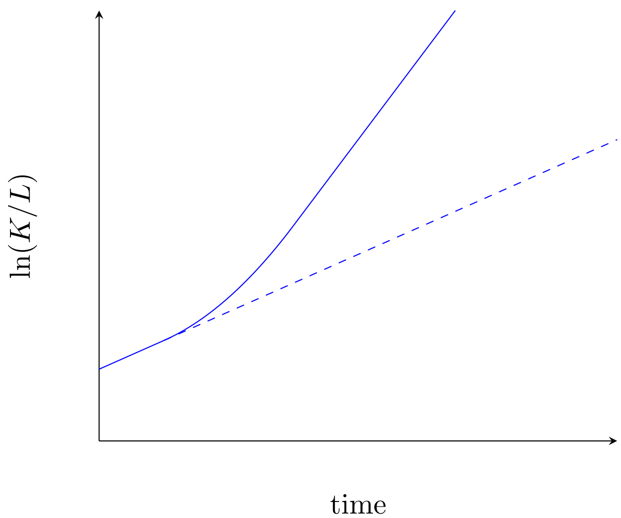

Problem 2.4 Assume production is Cobb-Douglas, $Y = K^{\alpha}(AL_Y)^{1-\alpha}$, with capital share $0 < \alpha < 1$ and depreciation δ. Population L grows at rate n. Assume productivity growth depends on research labor $L_A = s_R L$ and on existing productivity: $\dot{A} = \gamma L_A A^{\phi}$, where $\gamma > 0$, $0 < \phi < 1$, and $0 < s_R < 1$ are parameters. Production labor is $L_Y = (1 - s_R)L$.

(a) Derive the steady state growth rate of productivity. Show that per-capita output grows at the same rate.

To do this, take the dynamics of A, plug in for L_A and divide by A (to get an expression for the growth rate of A).

$$\dot{A} = \gamma L_A A^{\phi} \qquad \Longrightarrow \qquad \frac{\dot{A}}{A} = \frac{\gamma s_R L}{A^{1-\phi}}$$

In the steady state, we know that growth in A will be constant. Using the expression from above, we can see that the numerator is growing at the rate of n in the steady state (because of the L term). Thus, in order for this growth to be "balanced out," the denominator must be growing at the same rate. Looking more closely at this, we can determine the growth rate of the denominator:

$$(1-\phi)ln(A) \qquad \Longrightarrow \qquad (1-\phi)\frac{\dot{A}}{A} = (1-\phi)g_A,$$

where we can let g_A be a stand-in parameter for the steady state growth rate of A (which we are trying to figure out. In any regard, we know that the growth rate of the denominator equals n in the steady state, thus we have

$$(1-\phi)g_A = n \qquad \Longrightarrow \qquad g_A = \frac{n}{1-\phi}$$

Now we can turn to per-capita output. First, let's begin by taking the production function and dividing by L.

$$\frac{Y}{L} = \frac{K^{\alpha}(AL_Y)^{1-\alpha}}{L} \quad \Longrightarrow \quad ln\left(\frac{Y}{L}\right) = \alpha ln(K) + (1-\alpha)[ln(A) + ln(L_Y)] - ln(L)$$

$$ln\left(\frac{Y}{L}\right) = \alpha ln(K) + (1-\alpha)ln(A) + (1-\alpha)ln((1-s_R)L) - ln(L)$$

$$ln\left(\frac{Y}{L}\right) = \alpha ln(K) + (1-\alpha)ln(A) + (1-\alpha)ln(1-s_R) - \alpha ln(L)$$

Now take the time derivative of the expression.

$$\frac{\dot{Y/L}}{Y/L} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{A}}{A} - \alpha\frac{\dot{L}}{L}$$

$$= \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{n}{1-\phi} - \alpha n$$

$$= \alpha\left[\frac{\dot{A}}{A} + \frac{\dot{L}}{L} + \frac{\dot{k}}{k}\right] + (1-\alpha)\frac{n}{1-\phi} - \alpha n \qquad \text{(note that } K = ALk\text{)}$$

$$= \alpha\left[\frac{n}{1-\phi} + n + 0\right] + (1-\alpha)\frac{n}{1-\phi} - \alpha n \qquad \text{(invoking the steady state)}$$

$$= \frac{n}{1-\phi}$$

That is, we can see that output per-capita is growing at the same rate as productivity in the steady state.

(b) Suppose at time $t = 0$, the share of research labor is increase permanently to $\hat{s}_R > s_R$. Graph the time paths of productivity and of per-capita output. Is everyone better off with more research labor? Explain.

Here, in this richer model, there will be two effects happening simultaneously. More researchers means that there will be more productivity growth, but also less output because there are less people producing. To analyze the effects, let's put things in per-capita terms.

$$y = \frac{Y}{AL} = \frac{K^\alpha (A(1-s_R)L)^{1-\alpha}}{AL} = (1-s_R)^{1-\alpha} k^\alpha$$

$$\dot{k} = s(1-s_R)^{1-\alpha} k^\alpha - (\delta + n + g_A)k$$

Letting g_A denote the growth rate of productivity, you can go ahead and see that the steady state quantities of k, y, and c are

$$k^* = (1-s_R)\left(\frac{s}{\delta+n+g_A}\right)^{\frac{1}{1-\alpha}}$$

$$y^* = (1-s_R)\left(\frac{s}{\delta+n+g_A}\right)^{\frac{\alpha}{1-\alpha}}$$

$$c^* = (1-s)(1-s_R)\left(\frac{s}{\delta+n+g_A}\right)^{\frac{\alpha}{1-\alpha}}$$

Note that, at the steady state, $g_A = n/(1-\phi)$. We can see right away that everyone will not be unambiguously better off if s_R were to increase, as consumption will necessarily drop. What about the dynamics? Notice from before, that we can write the dynamics of Y/L as follows

$$\frac{\dot{Y/L}}{Y/L} = \alpha\left[\frac{\dot{A}}{A} + \frac{\dot{L}}{L} + \frac{\dot{k}}{k}\right] + (1-\alpha)\frac{\dot{A}}{A} - \alpha\frac{\dot{L}}{L}$$

$$= \frac{\dot{A}}{A} + \alpha\frac{\dot{k}}{k}$$

$$= \frac{\dot{A}}{A} + \alpha\left[s\left(\frac{1-s_R}{k}\right)^{1-\alpha} - \frac{\dot{A}}{A} - \delta - n\right]$$

$$= (1-\alpha)\frac{\dot{A}}{A} + \alpha\left[s\left(\frac{1-s_R}{k}\right)^{1-\alpha} - \delta - n\right]$$

We can see that the second term will decrease just after the change in s_R, but what about the first term?

$$\left.\frac{\dot{A}}{A}\right|_{t_0+\varepsilon} - \left.\frac{\dot{A}}{A}\right|_{t_0-\varepsilon} = \frac{\gamma L}{A^{1-\phi}}[\hat{s}_R - s_R] > 0$$

Thus, the growth rate of A will increase after the change. We know, when the steady state is reached again, that the growth rate of A will equal $n/(1-\phi)$, so in the interim growth will decline steadily. Now that we know how A growth responds (i.e. positively), we can see that the change in the growth of Y/L is ambiguous. While productivity growth has increased, there will be a slowdown in capital growth (in efficiency units) that offsets some (or possibly all) of that added growth. The distinction will depend on parameter estimates.

*Credit to several past TAs for some parts of these solutions.