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ECON 204A Midterm Solutions

Fall 2018

This exam is closed book. Most points are given for the correct set-up of a problem and for economically insightful interpretations. You have 75 minutes for a maximum score of 70 points.

Problem 1 (35p)
Consider a Solow model in which labor supply $H = h \cdot L$ is the product of population (L) and work hours per person (h). Work hours h are constant unless otherwise noted. Population L grows at rate n, productivity A grows at rate g, and the savings rate s is constant.

Production $Y = F(K, AH) = F(K, hAL)$ has constant returns to scale, has positive and declining marginal products, and satisfied the Inada conditions.

The capital K follows the differential equation $dK/dt = sY - \delta K$. Capital is assumed to depreciate more quickly when it's worked for more hours: $\delta = \delta_0 + \delta_1 \cdot h$ where $\delta_0, \delta_1 > 0$ are exogenous.

a. (10p) Derive a differential equation for the capital-labor ratio $k = K/(hAL)$. Derive a condition for the steady state value k^* and explain why k converges to a steady state value k^* from any non-zero starting value. Show that k^* is decreasing in h.

Starting with the definition of k, take logs and then take a time derivative.

$$k = \frac{K}{hAL} \implies ln(k) = ln(K) - ln(A) - ln(h) - ln(L) \implies \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \underbrace{\frac{\dot{A}}{A}}_{g} - \underbrace{\frac{\dot{L}}{L}}_{n}$$

Next, note that capital evolves according to $\dot{K} = sY - (\delta_0 + \delta_1 h)K$, where $Y = F(K, AL)$ is CRS. Plugging this into the last expression above:

$$\frac{\dot{k}}{k} = \frac{sY - (\delta_0 + \delta_1 h)K}{K} - g - n$$

$$\frac{\dot{k}}{k} = \frac{sy}{k} - (\delta_0 + \delta_1 h) - g - n$$

$$\dot{k} = sf(k) - (\delta_0 + \delta_1 h + n + g)k$$

Which is our differential equation for k. Next, to solve for the steady state value k^* we set $\dot{k} = 0$, which gives us our steady state condition:

$$(\delta_0 + \delta_1 h + n + g)k^* = sf(k^*)$$

Now, the following is the logic behind why k converges to k^* for any non-zero starting value (k_0). Because $f(k)$ is concave and $(g+n+\delta)$ is linear, there will be a single non-zero steady-state. Note that if $k_0 > k^*$, then $\dot{k} < 0$, and if $k_0 < k^*$, then $\dot{k} > 0$, implying that capital converges to k^* from any k_0. Finally, to show that k^* is decreasing in h, we will take a total derivative:

$$sf'(k^*)dk^* = (\delta_0 + \delta_1 h + n + g)dk^* + \delta_1 k^* dh$$

after some algebra we get:

$$\frac{dk^*}{dh} = \frac{k^*\delta_1}{sf'(k) - (\delta + n + g)} < 0$$

To see why this derivative is negative note the following:

$$sf'(k^*) - (\delta + n + g) < 0$$

$$s\frac{\alpha_k(k^*)f(k)}{k^*} - (\delta + n + g) < 0 \qquad \text{(using the definition of } \alpha(k))$$

$$\alpha_k(k^*)(\delta + n + g) - (\delta + n + g) < 0 \qquad \text{(using } sf(k^*) = (\delta + n + g)k^*)$$

The last line follows because $\alpha(k) \in (0,1)$. Thus we can see that the numerator of the derivative is strictly positive, while the denominator is negative.

b. (10p) Let Y^*/L denote output per person along a balanced growth path. Explain how Y^*/L depends on h and on other parameters. Derive a condition under which marginal increase in h will shift the balanced growth path up (i.e., a condition for $d(Y^*/L)/dh > 0$).

To begin, note that along a balanced growth path:

$$(Y^*/L) = Ahf(k^*)$$

Next, to simplify our analysis we can take logs:

$$ln(Y^*/L) = ln(A) + ln(h) + ln(f(k^*))$$

Now, to study how this depends on h we must take derivatives:

$$\frac{dln(Y^*/L)}{dh} = \frac{1}{h} + \frac{f'(k^*)}{f(k^*)}\frac{dk^*}{dh}$$

Plugging in our result from part a.:

$$\frac{dln(Y^*/L)}{dh} = \frac{1}{h} + \frac{f'(k^*)}{f(k^*)}\left(\frac{k^*\delta_1}{sf'(k) - (\delta + n + g)}\right)$$

Which we can rewrite as:

$$\frac{dln(Y^*/L)}{dh} = \frac{1}{h}\left(1 + \frac{\alpha_k(k^*)h\delta_1}{sf'(k) - (\delta + n + g)}\right)$$

Now, recalling our results from part a., it becomes clear that this derivative is positive when:

$$\frac{\alpha_k(k^*)h\delta_1}{sf'(k^*) - (\delta + n + g)} > -1$$

Or, to simplify this a bit:

$$\alpha_k(k^*)\delta_1 h < (\delta + n + g) - sf'(k^*)$$

Which we can rewrite as:

$$\alpha_k(k^*)\delta_1 h < (1 - \alpha_k(k^*))(\delta + n + g)$$

c. (15p) Suppose the economy is on the balanced growth path with parameters $\{h = 1, s = 0.2, n = g = 1\%, \delta_0 = \delta_1 = 2\%\}$ and has a capital share of 1/3. At time $t = t_0$, work hours increase by 10% to $h = 1.1$. Describe the impact on output per person (Y/L) over time. [Hint: Distinguish Y/L from Y^*/L and graph them on a log-scale.]

First, note what immediately happens to Y/L. Looking at our production function, it becomes clear that a 10% increase in h will in increase Y/L by $(1 - \alpha_k(k^*)) \cdot 10\% = 6.77\%$. However, we must also check the long run effects. Before solving for the change in the balanced growth path, it will be useful to perform a substitution in the denominator of the formula for $\frac{dln(Y^*/L)}{dh}$. Namely, recall from part a. that:

$$sf'(k^*) - (\delta + n + g) = \alpha_k(k^*)(\delta + n + g) - (\delta + n + g)$$

Thus:

$$\frac{dln(Y^*/L)}{dh} = \frac{1}{h}\left(1 + \frac{\alpha_k(k^*)h\delta_1}{\alpha_k(k^*)(\delta + n + g) - (\delta + n + g)}\right)$$

Plugging in the given values:

$$\frac{dln(Y^*/L)}{dh} = \frac{1}{1}\left(1 + \frac{\frac{1}{3}(.02)}{\frac{1}{3}(.06) - (.06)}\right) = \frac{5}{6}$$

Thus, we know that the new balanced growth path will be $\frac{5}{6} \cdot 10\% = 8.33\%$ higher than the previous one.

Problem 2 (35p)

This question is about the benefits of international linkages. Assume there are two economies, country $i = 1$ and country $i = 2$, that initially operate in isolation. The countries have the same Cobb-Douglas technology with capital share α , the same savings rate s, the same population growth n, and same depreciation rate δ . Their initial productivity levels A_i , population sizes L_i , and capital stocks K_i may differ. For parts (a) and (b), productivity growth g is exogenous and common; in (c) productivity will be endogenous.

a. (5p) Show that the capital-labor ratios in both countries converge to the same steady state value.

Starting with the definition of k_i, take logs and then take a time derivative.

$$k_i = \frac{K_i}{A_i L_i} \qquad \Longrightarrow \qquad ln(k_i) = ln(K_i) - ln(A_i) - ln(L_i) \qquad \Longrightarrow \qquad \frac{\dot{k}_i}{k_i} = \frac{\dot{K}_i}{K_i} - \underbrace{\frac{\dot{A}_i}{A_i}}_{g} - \underbrace{\frac{\dot{L}_i}{L_i}}_{n}$$

Next, note that capital in both economies evolves according to $\dot{K}_i = sY_i - \delta K_i$, where $Y_i = F(K_i, A_i L_i)$ is CRS. Plugging this into the last expression above (and then multiplying both sides by k_i), gives us the dynamics of capital in effective unit terms.

$$\dot{k}_i = sf(k_i) - (\delta + n + g)k_i \qquad\qquad \dot{k}_i = sk_i^{\alpha} - (\delta + n + g)k_i$$

The last statement is when production is assumed to be Cobb-Douglas. In the steady state, we know that $\dot{k}_i = 0$. Thus we can determine that

$$0 = sk_i^{*\alpha} - (\delta + n + g)k_i^*$$

$$sk_i^{*\alpha} = (\delta + n + g)k_i^*$$

$$k_i^{*1-\alpha} = \frac{s}{\delta + n + g}$$

$$k_i^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}.$$

Note that k_i^* does not depend on any economy-unique variables. Thus it is the same for both economies.

b. (10p) Suppose both economies are on their respective balanced growth paths.

i. Show that the return to capital $r = F_K - \delta$ is the same in both economies.

Following a similar procedure as in part a.:

$$r_i = F_K - \delta$$

Plugging in for each production function we get:

$$r_i = \alpha K_i^{\alpha-1} (A_i L_i))^{1-\alpha} - \delta = \alpha k_i^{\alpha-1} - \delta$$

Thus, in the steady state we have:

$$\alpha (k_i^*)^{\alpha-1} - \delta = \alpha \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha-1}{1-\alpha}} - \delta$$

Which simplifies to:

$$r_i = \alpha\left(\frac{\delta + n + g}{s}\right) - \delta$$

And again, we can see that this is the same for both economies.

ii. Suppose the two economies merged into an economic union with total capital $K_u = K_1 + K_2$, labor $L_u = L_1 + L_2$, and average productivity $A_u = \frac{A_1L_1+A_2L_2}{L_1+L_2}$. Show that output in the union, $Y_u = K_u^\alpha \cdot (A_u L_u)^{1-\alpha}$, equals the sum of the outputs $Y_1 + Y_2$ in the two separate economies.

We want to show that:

$$Y_u = Y_1 + Y_2$$

First note that in the steady state we have:

$$Y_u = A_u L_u y_u^*$$

Furthermore we know that:

$$y_u^* = (k^*)^\alpha$$

Plugging in our earlier results we get:

$$\begin{aligned}
Y_u &= \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} \cdot A_u L_u \\
&= \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{A_1L_1 + A_2L_2}{L_1+L_2}\right)(L_1+L_2) \\
&= \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} \cdot (A_1L_1 + A_2L_2) \\
&= y_1^* A_1 L_1 + y_2^* A_2 L_2 \\
&= Y_1 + Y_2
\end{aligned}$$

Note the fourth equality relies on the previously proven fact that $k_1^* = k_2^*$. Thus, the steady state output of the union economy is equal to the sum of the output of the individual economies in the steady state.

c. (20p) Suppose productivity growth in each country is the result of $R\&D$: $\dot{A}_i = \gamma \cdot L_{Ai}^\lambda$ where $L_{Ai} = s_R L_i$ is a share of the labor force, $\gamma > 0$ and $0 < \lambda \leq 1$. Labor $L_{Yi} = (1 - s_R)L_i$ goes into production.

i. Show that in both countries, productivity growth converges to the same constant g_A^*.

As before, we will begin by solving for productivity growth generally:

$$\dot{A}_i = \gamma L_{Ai}^\lambda \qquad \Longrightarrow \qquad \frac{\dot{A}_i}{A_i} = \frac{\gamma (s_R L_i)^\lambda}{A_i}$$

In the steady state, we know that growth in A will be constant. Using the expression from above, we can see that the numerator is growing at the rate of $\lambda \cdot n$ in the steady state (because of the L term). Thus, in order for this growth to be "balanced out," the denominator must be growing at the same rate. We know the growth rate of the denominator must be:

$$\frac{\dot{A}_i}{A_i} = g_{Ai},$$

where we can let g_A be a stand-in parameter for the steady state growth rate of A (which we are trying to figure out. In any regard, we know that the growth rate of the denominator equals $\lambda \cdot n$ in the steady state, thus we have

$$g^*_{Ai} = \lambda \cdot n$$

Which we can see, is the same in both economies.

ii. Show that regardless of starting positions, the economy with greater initial population will eventually have higher output per person than the economy with smaller population.

First note that along the balanced growth path:

$$\lambda n = \frac{\gamma s_R^\lambda L_i^\lambda}{A_i}$$

Thus:

$$A_i = \frac{\gamma s_R^\lambda}{\lambda n} \cdot L_i^\lambda$$

Therefore we can see that, as each parameter value is the same across the economies, the economy with the higher population (L_i) will have the higher A_i. Now, turning to the output per person equation:

$$Y_i^*/L_i = A_i f(k_i^*)$$

We can see that, as k^* and α are the same between the two economies, that the economy with the higher initial population (and consequently higher A_i) will have a higher output per person.

iii. Suppose both economies are on their respective balanced growth paths. At time $t = t_0$, they merge and pool their $R\&D$ efforts, so $\dot{A}_u = \gamma \cdot L_{Au}^\lambda$, where $L_{Au} = s_R L_u$. Explain why the newly merged economy will experience a period of high growth in output per person.

To begin note the following:

$$A_u^* = \frac{\gamma s_R^\lambda}{\lambda n} \cdot L_u^\lambda$$

What's more, note that because $L_u > L_i$, it must follow that $A_u > A_i$ and:

$$(Y_u^*/L_u) = A_u^*(k^*)^\alpha$$

Returning to our equation for the dynamics of output per person:

$$\frac{\dot{\overline{Y_u/L_u}}}{Y_u/L_u} = \frac{\dot{A}_u}{A_u} + \alpha \frac{\dot{k}_u}{k_u}$$

Plugging in our equation for $\dot{A}$:

$$\frac{\dot{\overline{Y_u/L_u}}}{Y_u/L_u} = \frac{\gamma \cdot (s_R L_u)^\lambda}{A_u} + \alpha \frac{\dot{k}_u}{k_u}$$

Now, let's begin to think about the dynamics of the newly merged economy. First note that at t_0 (when the economies merge), we know that $k_u^* = k_i^*$, thus k is already at the steady state.

When we turn to A_u, however, things get a bit more complicated. There are two possible assumptions one can make about A_u at t_0. One could assume that the new economy has the average productivity of the economies, or one could assume that the small economy adopts the productivity of the larger, making:

$$A_u = max\{A_1, A_2\}$$

No matter which assumption you use, however, we know that $A_u < A_u^*$ at t_0. Returning to our dynamics equation, this tells us that:

$$\frac{L_u^\lambda}{A_u} > \frac{L_u^\lambda}{A_u^*}$$

Which implies:

$$\frac{\dot{A}_u}{A_u} > g_A^*$$

and that there will be an upward kink in the time path of Y/L. We know that over time the high $\frac{\dot{A}_u}{A_u}$ will cause A_u to converge to A_u^*. What's more, we know that as A_u rapidly grows that $k_u < k^*$, however it will eventually return to the steady state.