Econ 204A: Section 8

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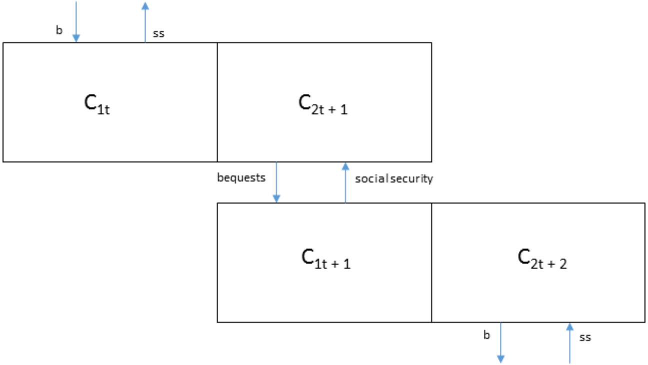
Diamond's Overlapping Generations Model

- Here, as with OG models generally, we will look at multiple generations interacting in an economy
- The basic model is cast in discrete time, with agents living for two periods, which we can think of a work period and a retirement period
- The two-period setup is not what is driving the results; it's just a simplification
- The important feature is the fact that there is turnover in the population: agents will "exit the model"
- Young / old coexistence is key, and ties the generations together over time; this framework allows for interesting insights into things like social security

Visually



Notation:

- 1: the young generation in some given period
- 2: the old generation in some given period
- t: time period

Basic Model

- Let's begin with a basic, motivating setup, which we will extend from
- Agents live for two periods: in the first period they work, and in the second period they are retired
- Starting with some arbitrary period t, young agents maximize the present value of lifetime consumption: $U = u(C_{1t}) + \beta u(C_{2t+1})$
- In the first period they inelastically supply 1 unit of time (or A_t units of labor; recall technology is labor augmenting) to earn income ($W_t = A_t w_t$) which can be consumed (C_{1t}) or saved (a_t)
- The savings in t (think of this happening at the end of the period) are used for productive purposes in $t+1$ and command an interest rate r_{t+1}
- In the second period, they consume their income from savings ($(1 + r_{t+1})a_t$)

- We can write the problem of the young as follows

$$\max_{C_{1t}, C_{2t+1}} U = u(C_{1t}) + \beta u(C_{2t+1}) \qquad \text{s.t.} \qquad C_{1t} + a_t = W_t$$

$$C_{2t+1} = (1 + r_{t+1}) a_t$$

- We can construct an intertemporal budget constraint by combining the two period-by-period constraints (solve for a_t in one and plug into the other)

$$\implies \qquad C_{1t} + \frac{1}{1 + r_{t+1}} C_{2t+1} = W_t$$

- The associated Lagrangian is given by

$$\mathcal{L} = u(C_{1t}) + \beta u(C_{2t+1}) + \lambda \left[W_t - C_{1t} - \frac{1}{1 + r_{t+1}} C_{2t+1} \right].$$

- The associated Euler Equation is then

$$\frac{u'(C_{1t})}{u'(C_{2t+1})} = \beta(1 + r_{t+1})$$

- The above optimality condition will imply a solution for optimal assets (plug in the period-by-period constraints in)

$$\frac{u'(W_t - a_t)}{u'((1 + r_{t+1})a_t)} = \beta(1 + r_{t+1})$$

- Concavity of u will imply that a is unique (can you prove this?)

- The savings rate is given by $s_t = a_t / W_t$ (note that this is not an exogenous savings rate!)
- Often times, to be explicit about the savings rate being a function of W_t and r_{t+1}, we'll write $s_t(W_t, r_{t+1})$
- Now that we've looked at the problem for the young generation (the workers), we can consider the problem of the old generation (the retirees)
- In this situation the old will simply consume what they have from their savings (they eat and then die)
- In the future we will consider things like taxes, social security transfers, incentives for bequests, etc. that make this a lot more interesting

Basic Model with Log / Power Utility

	Log	Power
Functional Form:	$u(C) = ln(C)$	$u(C) = \frac{C^{1-\gamma}}{1-\gamma}$
Euler Equation:	$C_{2t+1} = \beta(1 + r_{t+1})C_{1t}$	$C_{2t+1} = [\beta(1 + r_{t+1})]^{\frac{1}{\gamma}} C_{1t}$
Savings Function:	$s_t = \frac{\beta}{1+\beta}$	$s_t = 1 - \frac{1}{1 + \beta^{\frac{1}{\gamma}}(1 + r_{t+1})^{\frac{1}{\gamma-1}}}$

- For log utility the savings rate is constant; for power utility, the savings rate increases (decreases) with the interest rate if $\gamma < 1$ ($\gamma > 1$)
- The basic model leaves a lot to be desired; so far the generations aren't really "interacting" in any meaningful / insightful ways

Adding in a Production Sector

- The basic model we can consider as being *partial equilibrium*, it only considers the consumer's side of the market, and prices are simply given
- Here, we make it *general* by adding in a production sector, which will allow us to account for effects on prices that result from consumer choices
- Firms use the savings of the young to produce output $Y_t = F(K_t, A_t L_t)$, where F is CRS ($\implies y_t = f(k_t)$)
- Savings aggregate into capital according to $K_{t+1} = L_t a_t$ (Note: L_t is the number of workers in t)
- Assume each worker inelastically supplies one unit of labor when young, capital depreciates at a rate δ, A_t grows at rate g, and L_t grows at rate n

The Firm's Problem

$$\max_{k_t} \pi = f(k_t) - (\delta + r_t)k_t$$

- As written above, the firm maximizes its output net of depreciation and cost of capital
- Competitive determination of prices: capital and labor are paid their marginal product and there is no economic profits
  - $r_t = f'(k_t) - \delta$
  - $w_t = f(k_t) - k_t f'(k_t)$
- Remember that the savings at the end of the last period are used by firms today
  - the wage earned by the young today is linked to the savings decisions by the old yesterday

- The problem of the young / old generation will be very similar to what we had before
- The only thing we've done to the basic model is to close it and allow prices to adjust to behaviors
- This is the first step into getting the young / old to interact in economically meaningful ways
- Moving forward, we might want to think about the aggregate economy in equilibrium (steady state)

Steady State

- The steady state in this framework requires that $k_t = k_{t+1} = k^*$
- We can show this graphically by graphing k_t on the x-axis and k_{t+1} on the y-axis
- After deriving a relationship between k_{t+1} as a function of k_t (denoted by the correspondence $k_{t+1} = \mathbf{K}(k_t)$), the steady state will be given by any cross of the 45-degree line
- We'll have to be careful because there are a lot of moving parts in these general equilibrium models
  - possibility for the existence of multiple steady states
  - which, if any, are stable?

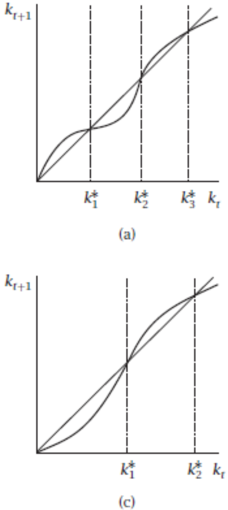


FIGURE 2.13 Various possibilities for the relationship between k_t and k_{t+1}

Take our expression for K_{t+1} from before and write it in terms of s_t.

$$K_{t+1} = L_t a_t \qquad \Longrightarrow \qquad K_{t+1} = L_t W_t s_t(W_t, r_{t+1})$$

Now we'll want to translate things into effective units because we know how some of these variables grow over time.

$$\overbrace{\frac{K_{t+1}}{A_{t+1}L_{t+1}}}^{k_{t+1}} = \frac{L_t}{L_{t+1}}\frac{W_t}{A_{t+1}} s_t(W_t, r_{t+1})$$

$$= \frac{L_t}{L_{t+1}}\frac{A_t}{A_{t+1}}\frac{W_t}{A_t} s_t(W_t, r_{t+1})$$

$$= \frac{L_t}{L_{t+1}}\frac{A_t}{A_{t+1}}\frac{\cancel{A_t} w_t}{\cancel{A_t}} s_t(A_t w_t, r_{t+1})$$

Now recall that $L_{t+1} = (1+n)L_t$ and $A_{t+1} = (1+g)A_t$.

$$k_{t+1} = \frac{s_t(A_t w_t, r_{t+1})}{(1+n)(1+g)} w_t$$

We know what w_t is though! (It's the income that isn't going to capital.)

$$k_{t+1} = \frac{s_t(A_t w_t, r_{t+1})}{(1+n)(1+g)} [f(k_t) - k_t f'(k_t)]$$

With specific functional forms, we can get k_{t+1} as an explicit function of k_t and graph it.

Ex: Log Utility and Cobb-Douglas Production

$$u(c) = ln(c) \qquad\qquad f(k) = k^{\alpha}$$

From before we found out what the savings rate would be with log utility (make sure you can actually derive this on your own).

$$s_t(A_t w_t, r_{t+1}) = \frac{\beta}{1+\beta}$$

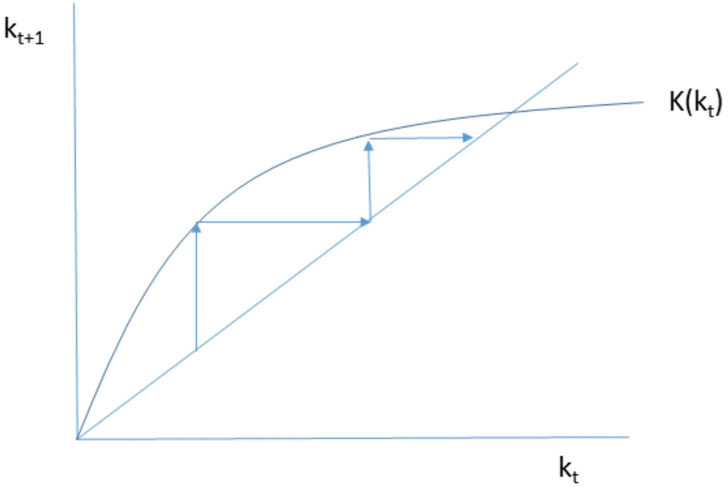
Again, notice that it's constant for this functional form. All that's left, then, is to find the wage.

$$w_t = f(k_t) - k_t f'(k_t) = k^{\alpha} - k_t(\alpha k_t^{\alpha-1}) = (1-\alpha)k_t^{\alpha}$$

Thus we'll have

$$k_{t+1} = \frac{(1-\alpha)\beta}{(1+\beta)(1+n)(1+g)} k_t^{\alpha}.$$

Graphically...



Extensions

We'll see a number of applications of this baseline. The OG model is very powerful / versatile and can capture a lot of nifty mechanisms.

- social security
- government debt / financing
- altruism and bequests
- human capital transfers across generations

Comment: Capital's / Labor's Share of Output

Here is just a short note on both capital's and labor's share.

Capital's Share: $$\alpha_K(k_t) = \frac{k_t f'(k_t)}{f(k_t)}$$

Labor's Share: $$\alpha_L(k_t) = \frac{f(k_t) - k_t f'(k_t)}{f(k_t)} = 1 - \alpha_K(k_t)$$

We can rewrite the factor payments as follows (which will sometimes be useful)

$$r_t = \frac{\alpha_K(k_t) f(k_t)}{k_t} - \delta \qquad w_t = \alpha_L(k_t) f(k_t).$$