E. Charlie Nusbaum
Econ 204A
December 1, 2017

Week 7 Section Notes

Problem 4.9. Consider an overlapping generations economy where individuals in generation t maximize utility

$$U = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

They work one unit when young, earn a wage w_t, save an amount a_t, and earn interest r_{t+1} on their savings. Ouput is produced with a Cobb-Douglas technology $Y_t = K_t^{\alpha} L_{t+1}^{1-\alpha}$ and there is 100% depreciation. The number of individuals in generation t is L_t. The size L_t of generations t grows at a fixed rate $n > 0$. The governmnet operates a pay-as-you go social security system as follows: Each period, the young pay a tax $T_{1t} = \tau w_t$, where $0 < \tau < 1$ (since labor supply is fixed, the tax is lump sum). The receipts are given to the old as a transfer $(-T_{2t})$.

a) Set up the individual optimization problem for generation t, derive the first order conditions, and solve for the asset accumulation as a function of T_{1t} and T_{2t+1}.

$$c_{1t} + a_t = (1-\tau)w_t = w_t - T_{1t} \qquad\qquad U = \ln(C_{1t}) + \beta \ln(C_{2t+1})$$

$$c_{2t+1} = (1+r_{t+1})a_t + \tau w_{t+1}(1+n) = (1+r_{t+1})a_t - T_{2t+1}$$

$$\Rightarrow c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - T_{1t} - \frac{T_{2t+1}}{1+r_{t+1}}$$

Here, we can either set up the Lagrangian or recognize that these preferences satisfy the hypothesis of DMRS. Thus, we can simply apply the tangency condition.

$$\frac{c_{2t+1}}{\beta c_{1t}} = 1 + r_{t+1} \qquad \Leftrightarrow \qquad c_{2t+1} = \beta(1+r_{t+1})c_{1t}$$

$$\Rightarrow (1+\beta)c_{1t} = w_t - T_{1t} - \frac{T_{2t+1}}{1+r_{t+1}} \qquad \Leftrightarrow \qquad \boxed{c_{1t} = \frac{1}{1+\beta}\left[w_t - T_{1t} - \frac{T_{2t+1}}{1+r_{t+1}}\right]}$$

$$\Rightarrow \qquad \boxed{c_{2t+1} = \frac{\beta(1+r_{t+1}))}{1+\beta}\left[w_t - T_{1t} - \frac{T_{2t+1}}{1+r_{t+1}}\right]}$$

$$\Rightarrow \qquad \boxed{a_t = \frac{1}{1+\beta}\left[\beta(w_t - T_{1t}) + \frac{T_{2t+1}}{1+r_{t+1}}\right]}$$

b) Explain why $\frac{-T_{2t+1}}{w_{t+1}} = \tau(1+n)$. Show how next period's capital-labor raio relates to the current capital-labor ration. [Hint: simplify $\frac{w_{t+1}}{1+r_{t+1}}$]

Because the population is growing from period to period, the transfers that you receive when you're old are being taken from a larger resource pool (i.e. τ draws from more wages). Thus, there is "more to go around" compared to the case with no population growth.

$$L_t T_{2t+1} = \tau w_{t+1} L_{t+1}$$

$$\Leftrightarrow \frac{T_{2t+1}}{w_{t+1}} = \tau \frac{L_{t+1}}{L_t}$$

$$\Leftrightarrow \boxed{\frac{T_{2t+1}}{w_{t+1}} = \tau(1+n)}$$

Next, let's turn to the mapping from $k_t \to k_{t+1}$.

$$K_{t+1} = L_t a_t$$

$$\Leftrightarrow L_{t+1}k_{t+1} = L_t \frac{1}{1+\beta}\left[\beta(w_t - \underbrace{\tau w_t}_{T_{1t}}) - \underbrace{\frac{1+n}{1+r_{t+1}}\tau w_{t+1}}_{T_{2t+1}}\right]$$

$$\Leftrightarrow k_{t+1} = \frac{1}{(1+n)(1+\beta)}\left[\beta(1-\tau)w_t - \frac{1+n}{1+r_{t+1}}\tau w_{t+1}\right]$$

$$\Leftrightarrow k_{t+1} = \frac{1}{(1+n)(1+\beta)}\left[\beta(1-\tau)(1-\alpha)k_t^\alpha - \frac{1+n}{\alpha k_{t+1}^{\alpha-1}}\tau(1-\alpha)k_{t+1}^\alpha\right]$$

$$\Leftrightarrow k_{t+1}\left(\frac{(1-\alpha)\tau + \alpha(1+\beta)}{\alpha(1+\beta)}\right) = \frac{\beta(1-\tau)(1-\alpha)}{(1+n)(1+\beta)}k_t^\alpha$$

$$\Leftrightarrow \boxed{k_{t+1} = \frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+n)[(1-\alpha)\tau + \alpha(1+\beta)}k_t^\alpha}$$

Notice that if $\tau = 0$, this reduces to the standard first order difference equation in these models.

c) Determine the steady state cpaital-labor ratio k^*. Show that k^* is decrease in τ.

In the steady state, $k_{t+1} = k_t = k^*$,

$$k^* = \frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+n)[(1-\alpha)\tau + \alpha(1+\beta)}k^{*\alpha}$$

$$\Leftrightarrow \boxed{k^* = \left(\frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+n)[(1-\alpha)\tau + \alpha(1+\beta)}\right)^{\frac{1}{1-\alpha}}}$$

Before taking the derivative, let's take a monotonic transformation:

$$\Leftrightarrow \ln(k^*) = \frac{1}{1-\alpha}\left[\ln\left(\frac{\alpha\beta(1-\alpha)}{(1+n)}\right) + \ln(1-\tau) - \ln((1-\alpha)\tau + \alpha(1-\beta))\right]$$

$$\Rightarrow \boxed{\frac{\partial \ln(k^*)}{\partial \tau} = -\frac{1}{1-\alpha}\left[\frac{1}{1-\tau} + \frac{1-\alpha}{(1-\alpha)\tau + \alpha(1-\beta)}\right] < 0}$$

Problem 4.10. Consider an overlapping generations economy with log-utility and Cobb-Douglas production. Individuals in generation t maximize utility

$$U = \ln(C_{1t}) + \beta \ln(C_{2t+1})$$

Individuals earn a wage w_t, save an amount a_t, and earn interest r_{t+1} on their savings. Output is $Y_t = K_t^{\alpha} L_{t+1}^{1-\alpha}$. Capital fully depreciates after one period. Generation t has L_t members. Population grows at a fixed rate n. The government may impose lum-sum taxes T_{1t} on the young and T_{2t} on the old. The government budget equation

$$D_{t+1} = (1 + r_t)D_t - [L_t T_{1t} + L_{t-1} T_{2t}]$$

describes the dynamics of government debt.

a) Set up the individual optimization problem for generation t, derive the first order condition, and solve for a_t as a function of w_t, r_{t+1}, T_{1t}, and T_{2t+1}.

$$c_{1t} + a_t = w_t - T_{1t} \qquad\qquad U = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$c_{2t+1} = (1 + r_{t+1})a_t - T_{2t+1}$$

$$\Rightarrow c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}} = w_t - T_{1t} - \frac{T_{2t+1}}{1 + r_{t+1}}$$

Now we can either set up the Lagrangian or we can recognize that these preferences satisfy the hypothesis of DMRS.

$$\frac{c_{2t+1}}{\beta c_{1t}} = 1 + r_{t+1}$$

$$\Rightarrow \boxed{c_{1t} = \frac{1}{1+\beta}\left[w_t - T_{1t} - \frac{T_{2t+1}}{1 + r_{t+1}}\right]}$$

$$\Rightarrow \boxed{a_t = \frac{1}{1+\beta}\left[\beta(w_t - T_{1t}) + \frac{T_{2t+1}}{1 + r_{t+1}}\right]}$$

b) Suppose the government is inactive so $T_{1t} = T_{2t} = D_t = 0\,\forall t$. Show how next period's capital-labor ratio depends on the current value of k_t. Show that the capital-labor ratio converges to a steady state k^*. Under what conditions about β is the economy dynamically inefficient?

$$D_{t+1} + K_{t+1} = L_t a_t$$

Impose $D_{t+1} = 0$

$$L_{t+1}k_{t+1} = L_t a_t$$

$$\Leftrightarrow k_{t+1} = \frac{1}{1+n}a_t$$

$$\Leftrightarrow k_{t+1} = \frac{1}{(1+n)(1+\beta)}\left[\beta((1-\alpha)k_t^\alpha - T_{1t}) - \frac{T_{2t+1}}{1+r_{t+1}}\right]$$

Impose $T_{1t} = T_{2t+1} = 0$

$$\Rightarrow \boxed{k_{t+1} = \frac{\beta(1-\alpha)}{(1+n)(1+\beta)}k_t^\alpha}$$

Because k_t^α is stricly concave, we know that $\exists!\ k^*$ such that $k^* = \frac{\beta(1-\alpha)}{(1+n)(1+\beta)}k^{*\alpha}$. Moreover, because $\frac{dk_{t+1}}{dk_t} \to \infty$ as $k_t \to 0^+$, we know that the unique steady state capital stock is on the interior of the state-space. In conjunction with the fact that $\frac{dk_{t+1}}{dk_t} \to 0$ as $k_t \to \infty$, we know that capital is increasing whenever $k_t < k^*$ and that capital is decreasing whenever $k_t > k^*$.

Note: We have not called them this, but steady states are effectively fixed points. That is they are values x such that $x = f(x)$. This will become very important next quarter in both Dynamic Programing and Game Theory.

Now, let's turn the conditions under which the economy is dynamically inefficient. Intuitively, the economy is dyanmically inefficient if we get less out of investing in capital than we put in. This leads to a natural check: the economy is dynamically efficient if $1 + r \geq \textit{growth rate}$. Here, that requires:

$$1 + r \geq 1 + n$$

Deriving steady state interest rate:

$$1 + r^* = \alpha k^{*\alpha-1}$$

$$\Leftrightarrow 1 + r^* = \alpha\left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{-1}$$

$$\Leftrightarrow 1 + r^* = (1+n)\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}$$

The dynamically efficient equation is satisfied if the second two terms are greater than 1

$$\frac{\alpha}{1-\alpha} \cdot \frac{1+\beta}{\beta} \geq 1$$

$$\Leftrightarrow \boxed{\beta \leq \frac{\alpha}{1-2\alpha}}$$

If you'd like a more rigorous explanation of why this is the case, see Romer p.g. 88-90

c) Suppose government debt carries into the next period that is proportional to the size of the yound generation, $D_{t+1} = d \cdot L_t$, for all t. where $d > 0$. Suppose $T_{2t} = 0$ and T_{1t} is set so that the budget equation is satisfied. Show how k_{t+1} depends on k_t and d. Derive k^*. Show that a higher d implies a lower k^*.

$$D_{t+1} = d \cdot L_t \qquad \textit{and} \qquad D_{t+1} = (1+r_t)D_t - L_t T_{1t}$$

$$\Rightarrow T_{1t} = \frac{(1+r_t) \cdot D_t}{L_t} - d$$

$$\Leftrightarrow T_{1t} = \frac{(1+r_t) \cdot d \cdot L_{t-1}}{L_t} - d$$

$$\Leftrightarrow T_{1t} = \left(\frac{1+r_t}{1+n} - 1 \right) d$$

Pluging this into the aggregate capital equation $K_{t+1} + D_{t+1} = L_t a_t$

$$L_{t+1}k_{t+1} + dLt = L_t \cdot \frac{\beta}{1+\beta} \left[w_t - \underbrace{\left(\frac{1+r_t}{1+n} - 1 \right) d}_{T_{1t}} \right]$$

$$\Leftrightarrow k_{t+1} + \frac{d}{1+n} = \frac{\beta}{(1+n)(1+\beta)} \left[(1-\alpha)k_t^{\alpha} + d - \frac{\alpha k_t^{\alpha-1}}{1+n} \cdot d \right]$$

$$\Leftrightarrow k_{t+1} = \frac{\beta}{(1+n)(1+\beta)} \left[(1-\alpha)k_t - \frac{\alpha \cdot d}{1+n} \right] k_t^{\alpha-1} - \frac{d}{(1+\beta)(1+n)}$$

Thus, k^* is given by the following equation (assuming no algebra mistakes!)

$$\boxed{k^* = \frac{\beta}{(1+n)(1+\beta)} \left[(1-\alpha)k^{*\alpha} - \frac{\alpha k^{*\alpha-1} \cdot d}{1+n} \right] - \frac{d}{(1+\beta)(1+n)}}$$

Now, let's see how k^* changes with d. The easiest way to proceed is to simply show that the k_{t+1} curve shifts downward.

$$\frac{\partial k_{t+1}}{\partial d} = -\frac{\alpha \cdot \beta \cdot k_t^{\alpha-1}}{(1+n)^2(1+\beta)} - \frac{1}{(1+\beta)(1+n)} < 0$$

Thus, an increase in d shifts down the $k_{t+1}(k_t)$ mapping and therefore k^*.

Note: Throughout this problem we I have implictly assumed that an interior solution exists.