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Problem Set 4

Romer Problem 2.1 Consider N firms each with the constant-returns-to-scale production function $Y = F(K, AL)$, or (using the intensive form) $Y = ALf(k)$. Assume $f'(\cdot) > 0$ and $f''(\cdot) < 0$. Assume that all firms can hire labor at wage wA and rent capital at cost r, and that all firms have the same value of A.

(a) Consider the problem of a firm trying to produce Y units of output at minimum cost. Show that the cost-minimizing level of k is uniquely defined and is independent of Y, and that all firms therefore choose the same value of k.

We can write the maximization problem for firms using a Lagrangian. The problem is to minimize costs subject to producing output Y.

$$\mathcal{L} = wAL + rkAL + \lambda[Y - ALf(k)]$$

$$\frac{\partial \mathcal{L}}{\partial L} = wA - \left(rkA - \frac{rK}{L}\right) + \lambda\left[\frac{f'(k)K}{L} - Af(k)\right] = 0$$
$$\implies \qquad w = \lambda[f(k) - kf'(k)]$$

$$\frac{\partial \mathcal{L}}{\partial k} = rAL - \lambda ALf'(k) = 0$$
$$\implies \qquad \lambda = \frac{r}{f'(k)}$$

Combining our results to remove λ, the necessary condition for optimality becomes

$$\frac{w}{r} = \frac{f(k) - kf'(k)}{f'(k)}.$$

To show that the solution to the above is unique, take a derivative of the RHS.

$$\frac{f'(k)[f'(k) - (f'(k) + kf''(k))] - f''(k)[f(k) - kf'(k)]}{f'(k)^2} = \frac{-f(k)f''(k)}{f'(k)^2} > 0$$

Since the sign of this derivative is always positive, it must be that the solution k that satisfies the necessary condition is unique. In addition, we can see directly that the solution to the above condition does not depend on the level of output, Y.

(b) Show that the total output of the N cost-minimizing firms equals the output that a single firm with the same production function has if it uses all the labor and capital used by the N firms.

The aim of this problem is to demonstrate that we only have to concern ourselves with an aggregate producer. Let's think about aggregating over a bunch of producers:

$$\sum_{i=1}^{N} Y_i = \sum_{i=1}^{N} AL_i f(k_i)$$

$$= Af(k) \sum_{i=1}^{N} L_i \qquad \text{(in (a) we found that } k_i = k \ \forall i\text{)}$$

$$= ALf(k).$$

Noting that, in the second step, we can utilize our insights from part (a) that all producers, regardless of their Y, will choose the same k (i.e. $k_i = k$). Last, because we are concerned about closed economies for now, if we add up the workers for each firm, we must recover the total population in the economy, L.

Romer Problem 2.2 *The elasticity of substitution with constant-relative-risk-aversion utility.* Consider an individual who lives for two periods and whose utility is given by

$$U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_2^{1-\theta}}{1-\theta}, \qquad \theta > 0, \qquad \rho > -1.$$

Let P_1 and P_2 denote the prices of consumption in the two periods, and let W denote the value of the individual's lifetime income; thus the budget constraint is $P_1C_1 + P_2C_2 = W$.

(a) What are the individual's utility-maximizing choices of C_1 and C_2, given P_1, P_2, and W?

Setting up a Lagrangian, taking first order conditions, and finding the Euler Equation ...

$$\mathcal{L} = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_2^{1-\theta}}{1-\theta} + \lambda\left[W - P_1C_1 - P_2C_2\right]$$

$$\frac{\partial \mathcal{L}}{\partial C_1} = C_1^{-\theta} - \lambda P_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial C_2} = \frac{C_2^{-\theta}}{1+\rho} - \lambda P_2 = 0$$

$$C_2 = \left(\frac{P_1}{(1+\rho)P_2}\right)^{\frac{1}{\theta}} C_1$$

We can plug in the EE into the budget constraint and solve for C_1, and then determine C_2 using the EE again.

$$P_1C_1 + P_2\left(\frac{P_1}{(1+\rho)P_2}\right)^{\frac{1}{\theta}} C_1 = W \implies \boxed{C_1 = \frac{W}{P_1 + P_2\left(\frac{P_1}{(1+\rho)P_2}\right)^{\frac{1}{\theta}}}}$$

$$\implies \boxed{C_2 = \left(\frac{P_1}{(1+\rho)P_2}\right)^{\frac{1}{\theta}} \frac{W}{P_1 + P_2\left(\frac{P_1}{(1+\rho)P_2}\right)^{\frac{1}{\theta}}}}$$

(b) The elasticity of substitution between consumption in the two periods is

$$-\frac{P_1/P_2}{C_1/C_2}\frac{\partial(C_1/C_2)}{\partial(P_1/P_2)} \qquad \text{or} \qquad -\frac{\partial ln(C_1/C_2)}{\partial ln(P_1/P_2)}.$$

Show that with the utility function from before, that the elasticity of substitution between C_1 and C_2 is $1/\theta$.

To start, let's take the EE that we found earlier and rewrite it (in part by taking logs).

$$\frac{C_2}{C_1} = \left(\frac{P_1}{(1+\rho)P_2}\right)^{\frac{1}{\theta}} \implies -ln\left(\frac{C_1}{C_2}\right) = \frac{1}{\theta}\left[ln\left(\frac{P_1}{P_2}\right) - ln(1+\rho)\right]$$

Now, simply take a derivative with respect to $ln(P_1/P_2)$, being mindful of that negative sign.

$$-\frac{\partial ln(C_1/C_2)}{\partial ln(P_1/P_2)} = \frac{1}{\theta}.$$

Romer Problem 2.4 Assume that the instantaneous utility function $u(C)$ in equation (2.1) is $ln(C)$. Consider the problem of a household maximizing (2.1) subject to (2.6). Find an expression for C at each time as a function of initial wealth plus the present value of labor income, the path of $r(t)$, and the parameters of the utility function.

First, write out the HH's problem and form a Lagrangian (Note: I eventually drop the limits of integration. All integrals are from $t = 0$ to ∞.).

$$\max U = \int_0^\infty e^{-\rho t} ln(C(t)) \frac{L(t)}{H} dt \qquad \text{s.t.} \qquad \int_0^\infty e^{-R(t)} C(t) \frac{L(t)}{H} \leq \underbrace{\frac{K(0)}{H} + \int_0^\infty e^{-R(t)} W(t) \frac{L(t)}{H} dt}_{\equiv \widetilde{W}}$$

$$\mathcal{L} = \int e^{-\rho t} ln(C(t)) \frac{L(t)}{H} dt + \lambda \left[\widetilde{W} - \int e^{-R(t)} C(t) \frac{L(t)}{H} \right]$$

$$\frac{\partial \mathcal{L}}{\partial C(t)} = \frac{e^{-\rho t}}{C(t)} \frac{L(t)}{H} - \lambda e^{-R(t)} \frac{L(t)}{H} = 0 \qquad \Longrightarrow \qquad C(t) = \frac{e^{-\rho t}}{\lambda e^{-R(t)}}$$

Now plug $C(t)$ into the budget constraint.

$$\widetilde{W} = \int e^{-R(t)} \frac{e^{-\rho t}}{\lambda e^{-R(t)}} \frac{L(t)}{H} dt \qquad \Longrightarrow \qquad \lambda = \frac{1}{\widetilde{W}} \int e^{-\rho t} \frac{L(t)}{H} dt$$

We can simplify this down even further. Note that, just like in Solow, $L(t) = L(0)e^{nt}$. Plug this into our expression for the shadow value.

$$\lambda = \frac{L(0)}{H} \frac{1}{\widetilde{W}} \int e^{-(\rho - n)t} dt$$

Notice that (if $\rho > n$, which we assume to be true), that the integral above can be solved: $\int_0^\infty e^{-(\rho-n)t} = 1/(\rho - n)$. And so

$$\lambda = \frac{L(0)}{H} \frac{1}{\widetilde{W}} \frac{1}{\rho - n}.$$

Plug this into the necessary condition for optimal consumption:

$$\boxed{C(t) = \frac{(\rho - n)\widetilde{W}}{L(0)/H} e^{R(t)-\rho t}}$$

Problem 3.1 Consider a household with a constant rate of time preference ρ who faces a given path of future real wages $w(t)$ and a constant interest rate r. The household supplies one unit of work ($L = 1$) and maximizes the utility function

$$U = \int_{t=0}^{T} e^{-\rho t} u(C(t)) dt,$$

where $u(C) = -e^{-\alpha C}$ and $\alpha > 0$ is a constant parameter (this is known as the exponential utility function with constant absolute risk aversion α). Initial asset holdings a_0 are given. Asset holdings must be non-negative at time T.

(a) Set up the Hamiltonian problem, apply the Maximum Principle, and derive an optimality condition for dC/dt.

$$\max U = \int_{0}^{T} e^{-\rho t} u(C(t)) dt \qquad \text{s.t.} \qquad \dot{a}(t) = r(t)a(t) + w(t) - C(t)$$

$$\mathcal{H}(C, a, \lambda, t) = e^{-\rho t} u(C) + \lambda \left[ra + w - C\right] \tag{1}$$

$$\frac{\partial \mathcal{H}}{\partial C} = 0: \qquad e^{-\rho t} u'(C) - \lambda = 0 \qquad \Longrightarrow \qquad \lambda = e^{-\rho t} u'(C) \tag{2}$$

$$\frac{\partial \mathcal{H}}{\partial a} = -\dot{\lambda}: \qquad \lambda r = -\dot{\lambda} \qquad \Longrightarrow \qquad \frac{\dot{\lambda}}{\lambda} = -r \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{a}: \qquad \dot{a} = ra + w - C \tag{4}$$

Where (1) is the Hamiltonian for this problem, and (2)-(4) are the results from the Maximum Principle. To get the Euler Equation, first take logs of (2) and then take a time derivative.

$$ln(\lambda) = -\rho t + ln\left(u'(C)\right) \qquad \Longrightarrow \qquad \frac{\dot{\lambda}}{\lambda} = -\rho + \frac{u''(C)}{u'(C)}\dot{C} \tag{5}$$

Next, equate (3) and (5).

$$-r = -\rho + \frac{u''(C)}{u'(C)}\dot{C}$$

We can utilize the functional form that we were given for u(C). Notice that

$$u'(C) = \alpha e^{-\alpha C} \qquad \text{and} \qquad u''(C) = -\alpha^2 e^{-\alpha C}$$

Thus we have

$$-r = -\rho - \alpha\dot{C} \qquad \Longrightarrow \qquad \boxed{\dot{C} = \frac{r-\rho}{\alpha}} \tag{6}$$

(b) Derive the intertemporal budget constraint and solve for the optimal consumption path $C(t)$. [Hint: a differential equation of the form $dx/dt = constant$ has the linear solution $x(t) = x(0) + t \times constant$.]

Before we get to the budget constraint, notice that (using the hint), we can solve (6).

$$C(t) = C(0) + \frac{r-\rho}{\alpha}t \tag{7}$$

Now we will derive the IBC in order to solve for $C(0)$. First, solve the differential equation given by (4). For a finite horizon problem with terminal period T, the general solution is given by

$$a(T)e^{-rT} = a(0) + \int_0^T \left[w(t) - C(t)\right]e^{-rt}dt \tag{8}$$

Our terminal condition states that $a(T) = 0$. The transversality condition necessitates that it not be positive (for optimality reasons), and we are told that a No Ponzi condition is in order (assets cannot be negative at the final date). That is $a(T) = 0$. We can plug this into (8) and then move some terms around . . .

$$\underbrace{a(0) + \int_0^T w(t)e^{-rt}dt}_{\equiv \widetilde{W}} = \int_0^T C(t)e^{-rt}dt. \tag{9}$$

Next we can plug (7) into (9) and simplify.

$$\begin{aligned}
\widetilde{W} &= \int_0^T \left[C(0) + \frac{r-\rho}{\alpha}t\right] e^{-rt}dt \\
&= \left[\frac{C(0)}{r}(1-e^{-rT}) + \int_0^T \frac{r-\rho}{\alpha}te^{-rt}dt\right] \\
&= \left[\frac{C(0)}{r}(1-e^{-rT}) + \left(\frac{r-\rho}{\alpha}\left[\frac{-te^{-rt}}{r}\right]_{t=0}^{t=T} - \frac{r-\rho}{\alpha}\int_0^T -\frac{e^{-rt}}{r}dt\right)\right] && \text{(int. by parts)} \\
&= \left[\frac{C(0)}{r}(1-e^{-rT}) - \frac{r-\rho}{\alpha}\frac{Te^{-rT}}{r} + \frac{r-\rho}{\alpha}\left(\frac{1-e^{-rT}}{r^2}\right)\right],
\end{aligned}$$

which simplifies to

$$C(0) = \frac{r\widetilde{W} - \frac{r-\rho}{\alpha}\left(\frac{1-(1+rT)e^{-rT}}{r}\right)}{1-e^{-rT}} \tag{10}$$

And so, finally, we have our optimal path for $C(t)$.

$$\boxed{C(t) = \frac{r\widetilde{W} - \frac{r-\rho}{\alpha}\left(\frac{1-(1+rT)e^{-rT}}{r}\right)}{1-e^{-rT}} + \frac{r-\rho}{\alpha}t}$$