Econ 204A
November 5, 2018

Problem Set 5

Romer Problem 2.6 *The productivity slowdown and saving.* Consider a Ramsey-Cass-Koopmans economy that is on its balanced growth path, and suppose there is a permanent fall in g.

(a) How, if at all, does this affect the $\dot{k} = 0$ curve?

The dynamics of capital are given by

$$\dot{k} = f(k) - c - (n + g)k.$$

The $\dot{k} = 0$ curve can therefore be written as

$$c = f(k) - (n + g)k. \tag{1}$$

Thus a decline in g will allow a higher level of consumption for every value of k. In other words, the $\dot{k} = 0$ curve will shift upward. Because of the concavity of $f(k)$, we know that the effect of a lower g on the curve defined by the above expression will be larger for larger k because g enters the expression linearly.

(b) How, if at all, does this affect the $\dot{c} = 0$ curve?

Recall that the Euler Equation governs the optimal time path of consumption and, in Romer's presentation, is given by

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho - \theta g}{\theta}.$$

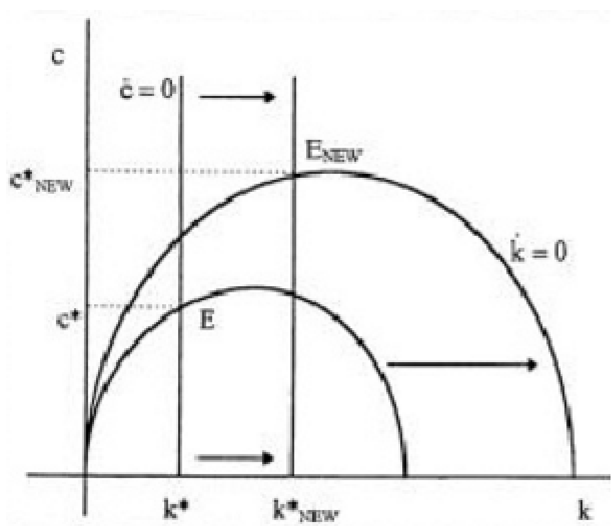
The $\dot{c} = 0$ curve can then be written as

$$f'(k^*) = \rho + \theta g. \tag{2}$$

Thus, a lower g would require (for the above to hold) that $f'(k^*)$ is lower. Because of declining marginal returns (i.e. $f''(k^*) < 0$) $f'(k^*)$ is lower when k^* is larger. Thus the $\dot{c} = 0$ curve will shift to the right.

(c) What happens to c at the time of the change?

Just like in the Solow model, the only time k can have discrete jumps is if there is something like an exogenous shock that (for example) destroys part of the capital stock. At the time of the change in g, therefore, there will be no jump in k. This is not true of c, however. When both of the curves shift, c will discretely jump accordingly onto the new saddle path. However, because both curves are shifting (and without values we don't know by how much these curves are shifting), we do not know if the new saddle path lies above or below (or possibly on top of) the old saddle path. Thus we cannot say for sure what will happen to c.[1]



(d) Find an expression for the impact of a marginal change in g on the fraction of output that is saved on the balanced growth path. Can one tell whether this expression is positive or negative?

First note that a balanced growth path implies that $\dot{k} = 0$ (effective unit variables aren't growing). Also note that, in this model, the savings rate on the balanced growth path will be given by the amount *not* consumed divided by total output:

$$s = \frac{f(k^*) - c^*}{f(k^*)}.$$

Recalling that we can rewrite the $\dot{k} = 0$ curve as done in (1), we can substitute in for the balanced growth path value of c with $c^* = f(k^*) - (n + g)k^*$ and simplify:

$$s = \frac{(n + g)k^*}{f(k^*)}.$$

[1]Credit for the image below goes to Romer's solutions.

We can now differentiate with respect to g.

$$\frac{ds}{dg} = \frac{f(k^*)[k^* + (n+g)\frac{dk^*}{dg}] - (n+g)k^*[f'(k^*)\frac{dk^*}{dg}]}{[f(k^*)]^2}$$

$$= \frac{f(k^*)k^* + (n+g)[f(k^*) - k^*f'(k^*)]\frac{dk^*}{dg}}{[f(k^*)]^2}$$

$$= \frac{f(k^*)k^* + (n+g)(1-\alpha(k^*))f(k^*)\frac{dk^*}{dg}}{[f(k^*)]^2}$$

$$= \frac{k^* + (n+g)(1-\alpha(k^*))\frac{dk^*}{dg}}{f(k^*)} \tag{3}$$

Note that in the above simplification I used the result that $\alpha(k) = kf'(k)/f(k)$. Now we'll need to figure out what $\frac{dk^*}{dg}$ is. Note that on the balanced growth path that $\dot{c} = 0$ and we can write this in terms of (2). We can then work out this derivative as follows.

$$f'(k^*) = \rho + \theta g \qquad \Longrightarrow \qquad f''(k^*)\frac{dk^*}{dg} = \theta \qquad \Longrightarrow \qquad \frac{dk^*}{dg} = \frac{\theta}{f''(k^*)} < 0 \tag{4}$$

Note that the derivative can be signed because we assume diminishing marginal returns on the production function. With this result, we can see that the derivative in (3) is ambiguous: the first term is positive and the second term is negative (the denominator is of course always positive). Thus a drop in g might increase, decrease, or keep the savings rate along the balanced growth path unchanged.

(e) For the case where the production function is Cobb-Douglas, $f(k) = k^\alpha$, rewrite your answer to part (d) in terms of ρ, n, g, θ, and α. (Hint: use the fact that $f'(k^*) = \rho + \theta g$.)

The first thing that we'll want to do is solve for k^*. When the economy is in steady state, we'll have that $\dot{c} = 0$. Using (1) with the Cobb-Douglas functional form we'll have that

$$\alpha k^{*\alpha-1} = \rho + \theta g \qquad \Longrightarrow \qquad k^* = \left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad f(k^*) = \left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{\alpha}{1-\alpha}}$$

We also know that, under Cobb-Douglas, $\alpha(k^*) = \alpha$. The only thing that's left is to determine $f''(k^*)$ for (4). Taking another derivative we can see that

$$f''(k^*) = \alpha(\alpha-1)k^{*\alpha-2} = \alpha(\alpha-1)\left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{\alpha-2}{1-\alpha}}.$$

Using this we can back out $\frac{dk^*}{dg}$.

$$\frac{dk^*}{dg} = \frac{\theta}{\alpha(\alpha-1)}\left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{2-\alpha}{1-\alpha}}$$

Plugging these results into (3) we have

$$\begin{aligned}
\frac{ds}{dg} &= \frac{\left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}} + (n+g)(1-\alpha)\frac{\theta}{\alpha(\alpha-1)}\left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{2-\alpha}{1-\alpha}}}{\left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{\alpha}{1-\alpha}}} \\
&= \left(\frac{\alpha}{\rho+\theta g}\right) - (n+g)\frac{\theta}{\alpha}\left(\frac{\alpha}{\rho+\theta g}\right)^2 \\
&= \frac{\alpha}{\rho+\theta g} - \frac{(n+g)\theta\alpha}{(\rho+\theta g)^2} \\
&= \frac{\alpha(\rho-\theta n)}{(\rho+\theta g)^2}, \qquad (5)
\end{aligned}$$

the sign of which is ambiguous and depends on the values of the parameters ρ, θ, and n.

Problem 3.2 Consider an economy with utility-maximizing, infinitely lived households; notation is as in Romer unless noted. Assume population (L) and the productivity index ($A = 1$) are constant. Preferences are

$$\max \int_{t=1}^{T} e^{-\rho t}u(C(t))dt,$$

where $u(\cdot)$ is increasing and concave and $0 < \beta < 1$. Capital accumulation is described by $\dot{k} = f(k) - c - \delta k$.

(a) Set up the Hamiltonian, apply the Maximum Principle, and derive a pair of equations that describe the dynamics of capital and consumption.

Note that $c = C$ because $A = 1$ and is constant.

$$\mathcal{H}(c, k, \lambda, t) = e^{-\rho t}u(c) + \lambda[f(k) - c - \delta k] \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial c} = 0 : \qquad \lambda = e^{-\rho t}u'(c) \tag{7}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} : \quad \frac{\dot{\lambda}}{\lambda} = -[f'(k) - \delta] = -r \tag{8}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{k} : \qquad \dot{k} = f(k) - c - \delta k \tag{9}$$

Take logs of (7) then differentiate w.r.t. time.

$$\begin{aligned} ln(\lambda) = -\rho t + \ln\left(u'(c)\right) \qquad \Longrightarrow \qquad \frac{\dot{\lambda}}{\lambda} &= -\rho + \frac{u''(c)\dot{c}}{u'(c)} \\ &= -\rho + \frac{u''(c)c}{u'(c)}\frac{\dot{c}}{c} \\ &= -\rho - \theta\frac{\dot{c}}{c} \end{aligned} \tag{10}$$

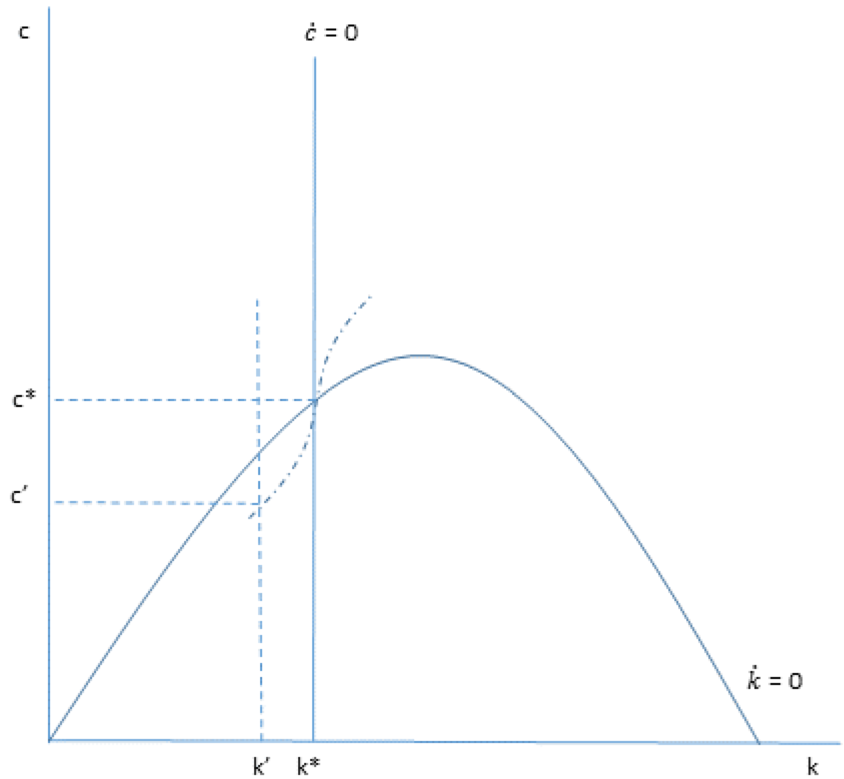
Next, equate (8) with (10) to get the Euler Equation.

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta} \tag{11}$$

Notice that I have plugged back in $r = f'(k) - \delta$ to make it explicit that consumption growth is a function of capital. Equations (9) and (11) describe the dynamics of capital and consumption.

(b) Suppose a hurricane destroys half of the capital stock. Describe graphically how consumption and the capital stock will adjust over time.

Notice that this sort of a shock does not shift either curve in the phase diagram. That is, the saddle path will remain unchanged. After the shock, the optimal choice would be to place the economy back on the saddle path at a lower level of consumption corresponding to whatever lower level of k the hurricane forced them to (denote these points c' and k'). Once on the saddle path, the economy will again return to the steady state along the saddle path.



(c) Starting in a steady state with positive depreciation, suppose depreciation is suddenly eliminated, $\delta = 0$. Describe graphically how consumption and the capital stock will adjust over time. Is there a finite steady state?

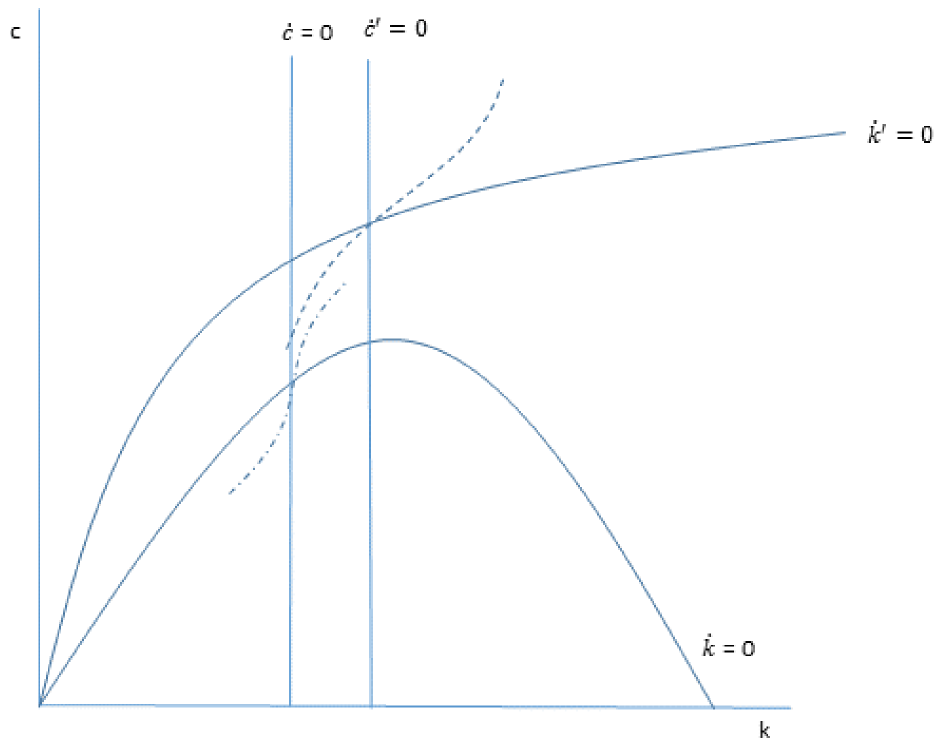
We should think about how this affects the $\dot{c} = 0$ and $\dot{k} = 0$ curves. First, the $\dot{c} = 0$ curve. From (11) the curve can be written as

$$f'(k^*) = \delta + \rho. \tag{12}$$

A reduction in δ to 0 thus means that this curve must shift to the right (again, recall that we assume that there is diminishing marginal returns). Perhaps more interesting is the effect that this has on the $\dot{k} = 0$ curve. From (9) we can write the curve as

$$c = f(k) - \delta k. \tag{13}$$

By eliminating δ, this curve no longer has the linear portion "pulling" $\dot{k}$ down. Thus the curve will now just increase indefinitely (though, at a decreasing rate because $f''(k) < 0$) and be higher (in the c dimension) for every point of k (except for the origin). Since both curves are shifting, we can't comment on whether the new saddle path will be higher or lower than before. Thus we can't say whether or not this change will lead to a discrete increase or decrease in consumption (k will not discretely change).



* Credit to several past TAs for some of these solutions.