Econ 204A

Problem Set 6

Romer Problem 2.9 *A closed-form solution to the Ramsey model.* (This follows Smith, 2006.) Consider the Ramsey model with Cobb-Douglas production, $y(t) = k(t)^\alpha$, and with the coefficient of relative risk aversion (θ) and capital's share (α) assumed to be equal.

(a) What is k on the balanced growth path (k^*)?

The $\dot{c} = 0$ curve directly pins down k^*.

$$f'(k^*) = \delta + \rho + \theta g$$

$$\alpha (k^*)^{\alpha - 1} = \delta + \rho + \theta g$$

$$k^* = \left(\frac{\alpha}{\delta + \rho + \theta g} \right)^{\frac{1}{1-\alpha}} \tag{1}$$

(b) What is c on the balanced growth path (c^*)?

We can combine the result from part (a) with the $\dot{k} = 0$ curve to obtain c^*.

$$c^* = k^{*\alpha} - (\delta + n + g)k^* = \left(\frac{\alpha}{\delta + \rho + \theta g} \right)^{\frac{\alpha}{1-\alpha}} - (\delta + n + g) \left(\frac{\alpha}{\delta + \rho + \theta g} \right)^{\frac{1}{1-\alpha}} \tag{2}$$

(c) Let $z(t)$denote the capital-output ratio, $k(t)/y(t)$, and $x(t)$ denote the consumption-capital ratio, $c(t)/k(t)$. Find expressions for $\dot{z}(t)$ and $\dot{x}(t)/x(t)$ in terms of z, x, and the parameters of the model.

First let's work with z. We can plug in for y, take logs, and then take a time derivative.

$$z = \frac{k}{k^\alpha} = k^{1-\alpha} \qquad \Longrightarrow \qquad ln(z) = (1-\alpha) ln(k) \qquad \Longrightarrow \qquad \frac{\dot{z}}{z} = (1-\alpha)\frac{\dot{k}}{k} \tag{3}$$

We can do something similar for x.

$$ln(x) = ln(c) - ln(k) \qquad \Longrightarrow \qquad \frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k} \tag{4}$$

Now, let's use the known expression for $\dot{k}/k$ for capital dynamics and plug it in to (3).

$$\frac{\dot{z}}{z} = (1-\alpha)\left[k^{\alpha-1} - \frac{c}{k} - (\delta+g+n)\right]$$

$$= (1-\alpha)\left[\frac{1}{z} - x - (\delta+g+n)\right] \tag{5}$$

Now we can do some more work on (4) by plugging in our known expressions for $\dot{k}/k$ and $\dot{c}/c$.

$$\frac{\dot{x}}{x} = \frac{\alpha k^{\alpha-1} - \delta - \rho - \theta g}{\theta} - \left[k^{\alpha-1} - \frac{c}{k} - (\delta+n+g)\right]$$

$$= \left(\frac{\alpha}{\theta} - 1\right) k^{\alpha-1} + \frac{c}{k} + \delta + n - \frac{\rho+\delta}{\theta}$$

$$= \left(\frac{\alpha}{\theta} - 1\right) \frac{1}{z} + x + \delta + n - \frac{\rho+\delta}{\theta}$$

$$= x + \delta + n - \frac{\rho+\delta}{\theta} \tag{6}$$

Recall that $\alpha = \theta$.

(d) Tentatively conjecture that x is constant along the saddle path. Given this conjecture:

i. Find the path of z given its initial value, $z(0)$.

First, rewrite (5) as follows.

$$\dot{z} = (1-\alpha)\left[1 - xz - (\delta+g+n)z\right]$$

$$= \underbrace{-(1-\alpha)[x+(\delta+g+n)]}_{\equiv A} z + \underbrace{(1-\alpha)}_{\equiv B}$$

$$= Az + B$$

Because x is *tentatively conjectured* to be constant along the saddle path, A and B are both constants. We can then easily solve the above differential equation (this case is classified as non-homogeneous with fixed coefficients).

$$\dot{z} = Az + B \qquad \Longrightarrow \qquad z = z(0)e^{At} + \frac{B}{A}(1 - e^{At}) \tag{7}$$

ii. Find the path of y given the initial value of k, $k(0)$. Is the speed of convergence to the balanced growth path, $dln[y(t) - y^*]/dt$, constant as the economy moves along the saddle path?

Rewrite consumption growth as $\dot{y}/y = \alpha(\dot{k}/k)$. Also note that we can express growth in capital in terms of growth in z (reworking the expression we gave in (3)),

$$\frac{\dot{k}}{k} = \frac{1}{1-\alpha}\frac{\dot{z}}{z}.$$

Putting this together we have

$$\frac{\dot{y}}{y} = \frac{\alpha}{1-\alpha}\frac{\dot{z}}{z} \qquad \Longrightarrow \qquad \frac{\dot{y}}{y} = \cancel{\frac{\alpha}{1-\alpha}(1-\alpha)}\left[\frac{1}{z} - x - (\delta + g + n)\right]$$

$$= \alpha\left[\frac{1}{z} - x - (\delta + g + n)\right] \tag{8}$$

We can solve the above differential equation (with variable coefficients). Define the *variable coefficient* in this situation as

$$\gamma(t) = \alpha\left[\frac{1}{z(t)} - x - (\delta + g + n)\right]$$

$$= \alpha\left[\frac{1}{z(0)e^{At} + \frac{B}{A}(1 - e^{At})} - x - (\delta + g + n)\right]$$

The solution to the differential equation thus becomes

$$y = y(0)e^{\int_0^t \gamma(s)ds} = k(0)^\alpha e^{\int_0^t \gamma(s)ds}. \tag{9}$$

Next we can look at the speed of convergence noted in the problem.

$$\frac{dln[y-y^*]}{dt} = \frac{\dot{y}}{y-y^*} = \frac{\alpha\left[\frac{1}{z} - x - (\delta+g+n)\right]k(0)^{\alpha}e^{\int_0^t \gamma(s)ds}}{k(0)^{\alpha}e^{\int_0^t \gamma(s)ds} - \left(\frac{\alpha}{\delta+\rho+\theta g}\right)^{\frac{\alpha}{1-\alpha}}} \tag{10}$$

Where $\dot{y}$ is obtained by combining (8) and (9). Notice that embedded in the solution is $\gamma(t)$, meaning that the speed of convergence is not constant as the economy moves along the saddle path.

(e) In the conjectured solution, are the equations of motion for c and k, (2.24) and (2.25), satisfied?

The solution will satisfy the equations of motion as they are derived from those equations.

Romer Problem 2.10 *Capital taxation in the Ramsey-Cass-Koopmans model.* Consider a Ramsey-Cass-Koopmans economy that is on its balanced growth path. Suppose that at some time, which we will call time 0, the government switches to a policy of taxing investment income at rate τ. Thus the real interest rate that households face is now given by $r(t) = (1-\tau)f'(k(t))$. Assume that the government returns the revenue it collects from this tax through lump-sum transfers. Finally, assume that this change in tax policy is unanticipated.

(a) How, if at all, does the tax affect the $\dot{c} = 0$ locus? The $\dot{k} = 0$ locus?

When households face this proportional tax on capital income, the effective rental rate on capital will decline. These taxes will affect the Euler Equation as it will reduce the return on capital which will affect the choice to consume or save. We'll have

$$\frac{\dot{c}}{c} = \frac{(1-\tau)f'(k) - \rho - \theta g}{\theta},$$

which implies the following $\dot{c} = 0$ curve.

$$f'(k^*) = \frac{\rho + \theta g}{1-\tau}.$$

Since τ is going from 0 to something strictly greater than zero, we know that (coupled with DMR) that k^* must decline. Notice, on the other hand, that the dynamics of capital won't change in response, as nothing has changed about the mechanics of capital accumulation. The $\dot{k} = 0$ curve is still given by

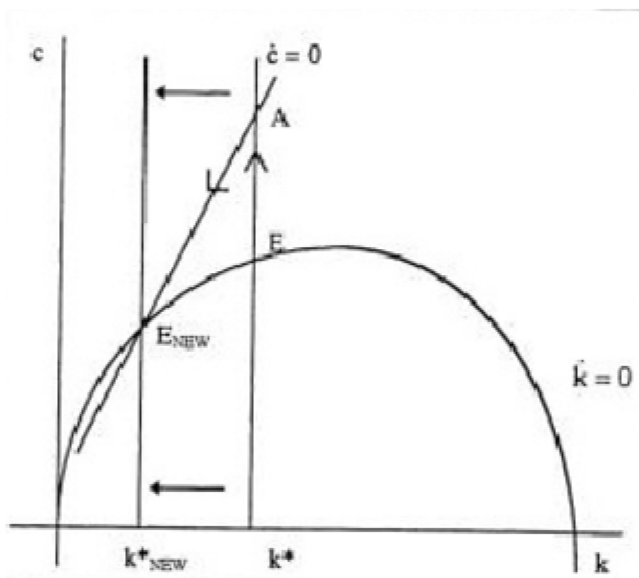
$$c = f(k) - (g+n)k.$$

Because tax revenues are rebated lump sum back to households, there will be no change in this curve.

(b) How does the economy respond to the adoption of the tax at time 0? What are the dynamics after time 0?

Assuming that the economy starts out on its balanced growth path. Because the adoption of the tax policy is unanticipated, agents will be unable to adjust behavior before the change. That is, the capital stock will be equal to the original k^*, which is higher than the new steady state capital stock.

Since there is only one curve shift, we know that the new saddle path will lie above the old saddle path. Optimizing agents will thus increase consumption at the time of the adoption of the tax policy to place them on the new path to the new steady state, which will ultimately be at a lower steady state value of c[1].



(c) How do the values of c and k on the new balanced growth path compare with their values on the old balanced growth path?

We know that the balanced growth levels of k and c will be lower (look at the image in the last problem). We can write out the expressions for k^* and c^*.

$$k^* = \left(\frac{\alpha(1-\tau)}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}} \qquad c^* = \left(\frac{\alpha(1-\tau)}{\rho+\theta g}\right)^{\frac{\alpha}{1-\alpha}} - (g+n)\left(\frac{\alpha(1-\tau)}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}}$$

The result that k^* will be lower is easy to see when τ is strictly positive. On c^* we must simply note that the second term (the negative term) will dominate the first term (the positive term) because

[1] Credit to the image below goes to Romer's solutions.

the factor $(1-\tau)$ is raised to a larger power.

(d) (This is based on Barro, Mankiw, and Sala-i-Martin, 1995.) Suppose there are many economies like this one. Workers' preferences are the same in each country, but the tax rates on investment income may vary across countries. Assume that each country is on its balanced growth path.

i. Show that the saving rate on the balanced growth path, $(y^* - c^*)/y^*$, is decreasing in τ.

$$s = \frac{y^* - c^*}{y^*} = \frac{f(k^*) - [f(k^*) - (g+n)k^*]}{f(k^*)} = \frac{(g+n)k^*}{f(k^*)} = (g+n)\frac{\alpha(1-\tau)}{\rho + \theta g} \tag{11}$$

Differentiating w.r.t. τ gives us a negative expression.

$$\frac{ds}{d\tau} = -\frac{g+n}{\rho + \theta g} < 0 \tag{12}$$

ii. Do citizens in low-τ, high k^*, high-saving countries have any incentive to invest in low-saving countries? Why or why not?

The purpose of investing in another country depends on the rate of return that agents can expect to command by doing so. Without loss of generality, let country A be the low-τ, high k^*, high-saving economy. Comparing the two interest rates, we want to know if one is higher. Recall that $(1-\tau)f'(k^*) = \rho + \theta g$.

$$(1-\tau_A)f'(k_A^*) \qquad ? \qquad (1-\tau_B)f'(k_B^*)$$

$$\rho + \theta g \qquad = \qquad \rho + \theta g$$

That is, we can see that there will be no incentive to invest in the other country along the steady state (note that this is assuming the technologies are the same in both countries).

(e) Does your answer to part (c) imply that a policy of subsidizing investment (that is, making $\tau < 0$), and raising the revenue for this subsidy through lump-sum taxes, increases welfare? Why or why not?

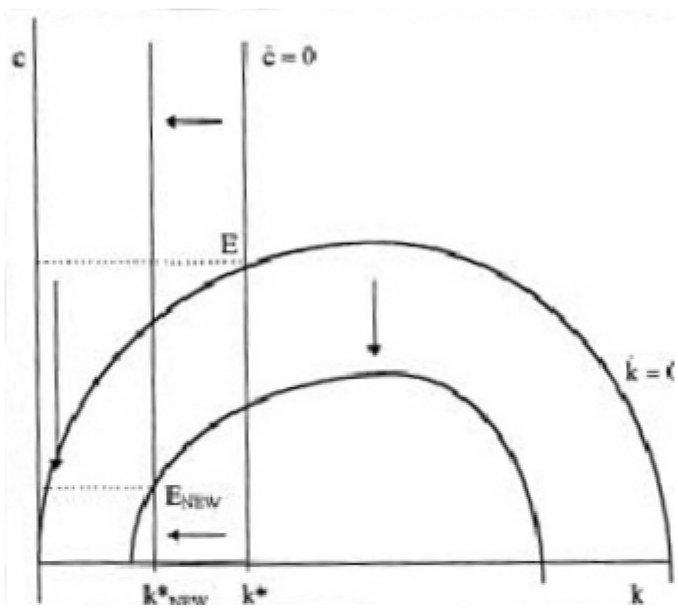
Since there are no externalities in this economy, the outcome with $\tau = 0$ is Pareto optimal. Another way to say this is that a social planner would select to maximize welfare. The result of having a negative tax rate (a subsidy) would shift the $\dot{c} = 0$ curve to the right (instead of the left). Optimizing agents would then have to discontinuously decrease their consumption in order to place themselves on the path to a higher steady state down the road. The loss resulting from this discrete drop today will outweigh (when discounted) the gain of higher balanced growth consumption in the future. At the least, though, the discontinuous drop places this sort of intervention outside the scope of possible Pareto improvements.

(f) How, if at all, do the answers to parts (a) and (b) change if the government does not rebate the revenue from the tax but instead uses it to make government purchases?

First we can rewrite the dynamics of k to capture the fact that the taxes are not being rebated to households.

$$\dot{k} = f(k) - c - (n+g)k - G$$

Relative to the expression before, the new dynamics are shifted down by G at ever point of k. As before, the τ on capital income shifts the $\dot{c} = 0$ curve. So the changes to our phase diagram will look like the following[2].



We cannot comment on whether or not consumption will increase or decrease because both curves are shifted.

[2]Credit again goes to Romer's solutions.

Problem 3.3 Consider a continuous-time representative agent economy with constant population $L = 1$ and constant productivity $A = 1$. Individuals have preferences

$$U = \int_{t=1}^{\infty} e^{-\rho t} u(C(t)) dt,$$

where u is increasing and concave. The stock of capital for the aggregate economy evolves according to the equation $\dot{K} = I - \delta K$. Government spending is a function of time $G(t)$. The national income identity is given by $Y = F(K, L) = C + I + G$, where F satisfies the Inada conditions. Taxes are lump sum and equal to government expenditure in every period. The representative agent expects that government expenditure will be constant over time.

(a) Set up the representative agent's problem. Apply the maximum principle.

Notice that in this problem $k = K$ and $c = C$.

$$\mathcal{H}(c, k, \lambda, t) = e^{-\rho t} u(c) + \lambda[f(k) - c - \delta k - G] \tag{13}$$

$$\frac{\partial \mathcal{H}}{\partial c} = 0 : \qquad \lambda = e^{-\rho t} u'(c) \tag{14}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} : \quad \frac{\dot{\lambda}}{\lambda} = -[f'(k) - \delta] \tag{15}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{k} : \qquad \dot{k} = f(k) - c - \delta k - G \tag{16}$$

(b) Derive the phase diagram and show that K converges to a steady state value K^* from any initial value K_0. Explain why C is an increasing function of K during the convergence process.

First let's find the Euler equation. We employ the usual steps of taking logs of (14) and differentiating w.r.t. t. Then we equate that result with (15) and simplify. (If you still need help with this please consult previous section slides and past problem sets.) The fruits of our labor produce

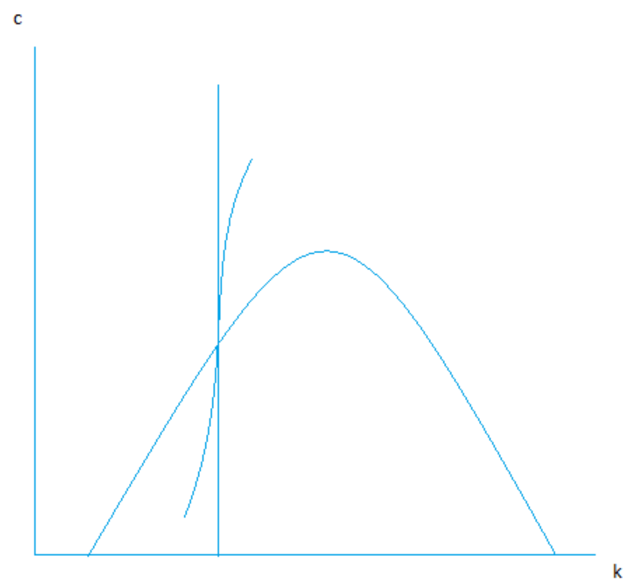
$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta}. \tag{17}$$

And so the dynamics are given by (16) and (17). The $\dot{c} = 0$ and $\dot{k} = 0$ curves are given by

$$f'(k^*) = \delta + \rho \tag{18}$$

$$c = f(k) - \delta k - G \tag{19}$$

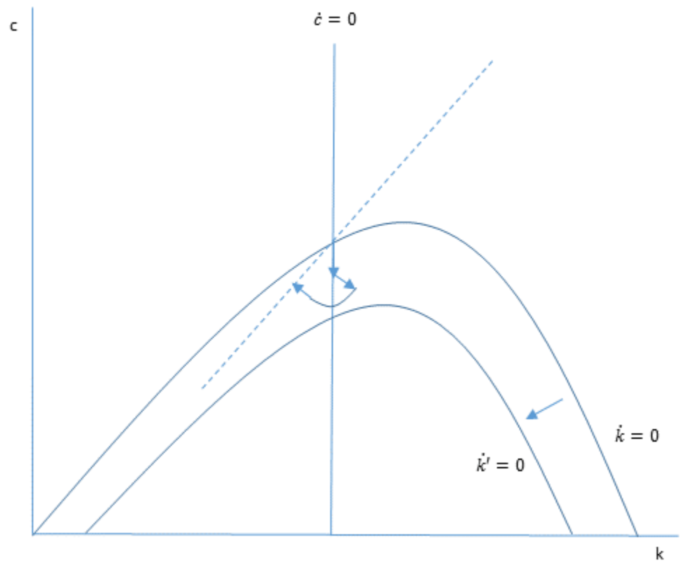
The associated phase diagram is given below.



For any given $K_0 > 0$, we'll get convergence. If k is small (less than k^*) we'll have that $\dot{k} > 0$ because of the Inada conditions. If k is large we'll have the opposite, and so there will be convergence. With regard to c, for low k we'll have that $f'(k)$ will be high and thus $\dot{c} > 0$. For high k we'll have the opposite. Couple this with the dynamics laid out with regard to convergence of capital, and we can see that c is an increasing function of k.

(c) Suppose at time $t = 0$, the government announces a tax-financed increase in government spending starting at $t = 1$ and ending at $t = 2$. Assuming the economy was at the steady state, show the dynamics of C, K, and the interest rate r. Illustrate these dynamics in the phase diagram and by sketching time-series charts for C, K, and r.

Agents know that the economy will return to the previous level of government spending after one period, and can thus plan for this in the future. Since the government is essentially "burning money," the $\dot{k} = 0$ curve drops and agents adjust their consumption immediately. When the increase in spending is announced, they adjust their consumption downward. This downward adjustment leads to an increase in k. In $t = 1$, the policy takes place and the $\dot{k} = 0$ curve shifts down. The quadrant that the economy is in puts downward pressure on k. When the economy crosses the $\dot{c} = 0$ curve eventually, the economy will begin moving upward and to the left. At $t = 2$ the $\dot{k} = 0$ curve shifts back, and if optimized properly, the economy should be on the original saddle path en route to the balanced growth path. This is shown below.



Problem 3.4 Consider an individual facing the following utility maximization problem,

$$\max \int_{t=0}^{\infty} e^{-\rho t} u(c(t), l(t)) dt,$$

where C is consumption and l is leisure; utility is increasing and concave in both arguments. The individual is endowed with 1 unit of time and works $n(t) = 1 - l(t)$. The per-worker capital stock evolves according to $\dot{k} = k^{\alpha} n^{1-\alpha} - c - G$, where G is a constant level of per-capita government spending. There is no depreciation and the population is constant.

(a) Assume for now that leisure l is constant at some level $0 < l < 1$. Set up the Hamiltonian and find the optimality conditions.

For this part the problem is basically what we have seen before.

$$\mathcal{H}(c, k, \lambda, t) = e^{-\rho t} u(C, l) + \lambda[k^{\alpha} n^{1-\alpha} - c - G] \tag{20}$$

$$\frac{\partial \mathcal{H}}{\partial c} = 0 : \qquad \lambda = e^{-\rho t} u_c(c, l) \tag{21}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} : \quad \frac{\dot{\lambda}}{\lambda} = -\alpha \left(\frac{n}{k}\right)^{1-\alpha} \tag{22}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{k} : \qquad \dot{k} = k^{\alpha} n^{1-\alpha} - c - G \tag{23}$$

Now we employ the usual steps. Take logs of (21) and time differentiate.

$$ln(\lambda) = -\rho t + ln\left(u_c(c,l)\right) \qquad \Longrightarrow \qquad \frac{\dot{\lambda}}{\lambda} = -\rho + \frac{u_{cc}(c)c}{u_c(c)}\frac{\dot{c}}{c} \qquad \Longrightarrow \qquad \frac{\dot{\lambda}}{\lambda} = -\rho - \theta\frac{\dot{c}}{c}$$

Equating this result with (22) and simplifying gives us the Euler Equation.

$$\Longrightarrow \qquad \frac{\dot{c}}{c} = \frac{1}{\theta}\left[\alpha\left(\frac{n}{k}\right)^{1-\alpha} - \rho\right] \tag{24}$$

The solution will thus satisfy (23) and (24).

(b) Solve for the steady state values of c^* and k^*. How do changes in G affect both? How do exogenous changes in l affect both?

First, k^* is pinned down from the $\dot{c} = 0$ curve. Set (24) equal to zero and solve for k (also recall that $n = 1 - l$).

$$k^* = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}(1-l) \tag{25}$$

We can then determine c^* by plugging this result into the $\dot{k} = 0$ curve implied by (23).

$$c^* = \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}(1-l)^{\alpha} - G \tag{26}$$

G does not affect k^*, which makes sense because it does not appear in the $\dot{c} = 0$ curve. It does affect c^* because it affects the $\dot{k} = 0$ curve. A higher G reduces c^*, as less money is available to consume at any level of k. With regard to l, a higher l reduces both k^* and c^*. This makes sense as more leisure reduces production which is used for both consumption and capital accumulation.

(c) Now let $l(t)$ be another choice variable. Set up the Hamiltonian and find the optimality conditions.

Now we have two choice variables!

$$\mathcal{H}(c,l,k,\lambda,t) = e^{-\rho t}u(c,l) + \lambda[k^{\alpha}(1-l)^{1-\alpha} - c - G] \tag{27}$$

$$\frac{\partial\mathcal{H}}{\partial c} = 0: \qquad \lambda = e^{-\rho t}u_c(c,l) \tag{28}$$

$$\frac{\partial\mathcal{H}}{\partial l} = 0: \qquad \lambda = \frac{e^{-\rho t}u_l(c,l)}{(1-\alpha)\left(\frac{k}{1-l}\right)^{\alpha}} \tag{29}$$

$$\frac{\partial\mathcal{H}}{\partial k} = -\dot{\lambda}: \quad \frac{\dot{\lambda}}{\lambda} = -\alpha\left(\frac{1-l}{k}\right)^{1-\alpha} \tag{30}$$

$$\frac{\partial\mathcal{H}}{\partial\lambda} = \dot{k}: \qquad \dot{k} = k^{\alpha}(1-l)^{1-\alpha} - c - G \tag{31}$$

(d) Assume utility has the form $u(c,l) = ln(c) + ln(l)$. Solve for the steady state values c^*, k^*, and n^*. How do changes in G affect c^*, k^*, and n^*?

First, let's use the functional form specification for the utility function to rewrite the optimality conditions.

$$\frac{\partial\mathcal{H}}{\partial c} = 0: \qquad \lambda = e^{-\rho t}\frac{1}{c} \tag{28'}$$

$$\frac{\partial\mathcal{H}}{\partial l} = 0: \qquad \lambda = \frac{e^{-\rho t}}{(1-\alpha)\left(\frac{k}{1-l}\right)^{\alpha} l} \tag{29'}$$

Combining both of these results, we get an optimality condition for the relationship between consumption and leisure today.

$$c = \frac{(1-\alpha)k^{\alpha}l}{(1-l)^{\alpha}} \tag{32}$$

The usual steps combining (28') and (30) will give us the Euler Equation.

$$\frac{\dot{c}}{c} = \alpha\left(\frac{1-l}{k}\right)^{1-\alpha} - \rho \tag{33}$$

In the steady state, we'll have $\dot{c} = 0$ and $\dot{k} = 0$. That is,

$$c = k^{\alpha}(1-l)^{1-\alpha} - G \tag{34}$$

$$\rho = \alpha \left(\frac{1-l}{k}\right)^{1-\alpha}. \tag{35}$$

The answer will be given by the solution to (32), (34), and (35). Now we must do a lot of arithmetic. Let's solve (35) for k^*.

$$k^* = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} (1-l^*) \tag{36}$$

Now plug the result from (32) for c and (36) in for k into (34).

$$\frac{(1-\alpha)l^*}{\cancel{(1-l^*)^{\alpha}}} \left(\left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \cancel{(1-l^*)^{\alpha}} \right) = \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} (1-l^*) - G$$

$$(1-\alpha)l^* = (1-l^*) - G\left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}$$

$$l^* = \frac{1}{2-\alpha}\left[1 - G\left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right] \tag{37}$$

And so, we have that

$$n^* = 1 - \frac{1}{2-\alpha}\left[1 - G\left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right] \tag{38}$$

$$k^* = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \left\{1 - \frac{1}{2-\alpha}\left[1 - G\left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right]\right\} \tag{39}$$

$$c^* = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \left\{1 - \frac{1}{2-\alpha}\left[1 - G\left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right]\right\} - G \tag{40}$$

We can see by how G enters the above expressions that an increase in G will increase n^* and k^*, but decrease c^* (notice that the positive term for G in (40) has a factor that is less than 1 on it; the $-G$ will thus dominate). That is, the government spending will induce more work and a higher capital stock at the cost of less consumption.