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Econ 204A

Problem Set 7

Problem 4.1 Consider an overlapping generations economy. Individuals in generation t maximize utility $U = ln(c_{1t}) + \beta ln(c_{2t+1})$. They earn a wage w_t when young, save an amount a_t, and earn interest rate at r_{t+1} on their savings. Technology is Cobb-Douglas with capital share α and 100% depreciation, so output net of depreciation is $Y_t = K_t^{\alpha} L_t^{1-\alpha} - K_t$. Wage and interest are determined competitively. There is no population growth and no productivity growth; population size is L.

(a) Set up the individual optimization problem for generation t; derive the savings function; state the equilibrium condition, and determine the steady state capital-labor ratio k^*.

We first want to setup the Lagrangian. To do so, we must derive the intertemporal budget constraint faced by the young at period t. The two period-by-period budget constraints are given by the following.

First Period: $c_{1t} + a_t = w_t$

Second Period: $c_{2t+1} = (1 + r_{t+1})a_t$

Solving for a_t in the first period's constraint, plugging the result into the second period, and reorganizing, we find that

$$c_{1t} + \frac{1}{1 + r_{t+1}} c_{2t+1} = w_t.$$

The Lagrangian and the associated FOCs are given by

$$\mathcal{L} = ln(c_{1t}) + \beta ln(c_{2t+1}) + \lambda \left[w_t - c_{1t} - \frac{1}{1 + r_{t+1}} c_{2t+1} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0: \qquad \lambda = \frac{1}{c_{1t}}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0: \qquad \lambda = \frac{\beta(1 + r_{t+1})}{c_{2t+1}}$$

$$\implies \qquad c_{2t+1} = \beta(1 + r_{t+1})c_{1t}$$

Plugging this into the budget constraint, we can determine both c^*_{1t}, c^*_{2t+1}, and a^*_t.

$$c^*_{1t} = \frac{w_t}{1+\beta} \qquad c^*_{2t+1} = \frac{\beta(1+r_{t+1})}{1+\beta} w_t \qquad a^*_t = \frac{\beta}{1+\beta} w_t$$

The savings function is given by $s_t = a^*_t / w_t$.

$$s_t = \frac{\beta}{1+\beta}$$

In equilibrium we know that the asset savings of the young equals the capital used by firms in producing.

$$K_{t+1} = L_t s_t W_t \qquad \Longrightarrow \qquad k_{t+1} = s_t w_t$$

Where the second step puts our variables into effective unites noting that L_t and A_t are not growing over time. Moving forward, we can plug in for both s_t and $w_t = f(k_t) - k_t f'(k_t) = (1-\alpha)k_t^\alpha$, where we utilize the fact that production is Cobb-Douglas.

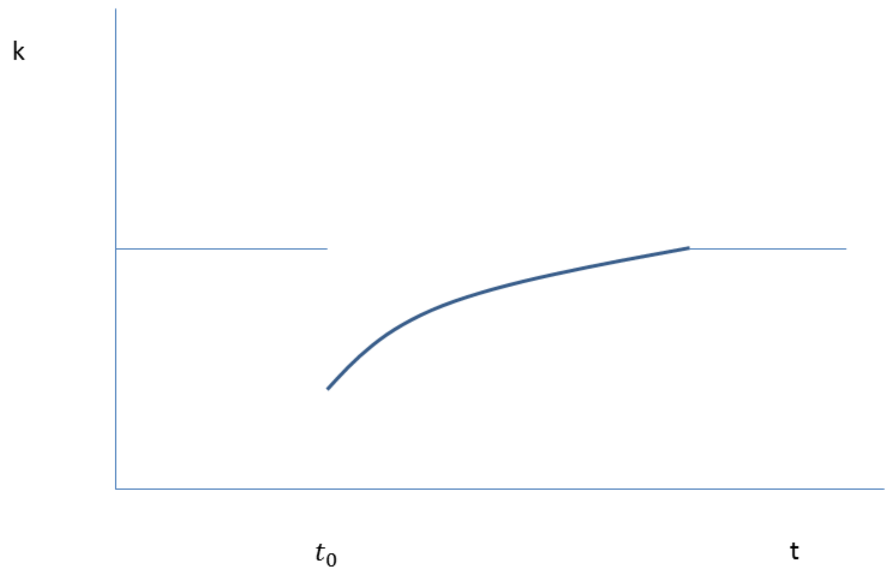
$$k_{t+1} = \frac{(1-\alpha)\beta}{1+\beta} k_t^\alpha$$

In the steady state, we have that $k_{t+1} = k_t = k^*$. We have

$$k^* = \left(\frac{(1-\alpha)\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}.$$

(b) Start in the steady state and assume dynamic efficiency. Suppose in some period t_0 that the old generation discovers that a fraction $\phi > 0$ of their savings was "invested" in assets that proved worthless (say, unfinished houses in the desert), so that the useful capital stock is only $K_{t_0} = (1-\phi)La_{t_0 - 1}$. Describe how the economy will evolve over time after this loss of wealth. Sketch a time path for k_t.

The change in the capital stock in t_0 doesn't alter any part of the steady state capital expression, and so won't affect the steady state level (i.e. the economy will eventually return to the same point). There will be a drop in k at t_0, and the economy will converge over time.



Problem 4.2 Consider an overlapping generations economy in which young individuals work full time (1 time unit) and old individuals work part-time, namely γ time units (where $0 < \gamma < 1$). The economy has a Cobb-Douglas technology with capital share α and 100% depreciation. Individuals of generation t maximize utility $ln(c_{1t}) + \beta ln(c_{2t+1})$, where $0 < \beta < 1$. There is no government, no population growth, and no productivity growth.

(a) Derive individual asset holdings a_t as a function of w_t, w_{t+1}, and r_{t+1}. What are the effects of second period work effort (γ) on the level and the interest sensitivity of savings? Provide an economic interpretation of these effects.

First we derive the intertemporal budget constraint.

First Period: $c_{1t} + a_t = w_t$

Second Period: $c_{2t+1} = (1 + r_{t+1})a_t + \gamma w_{t+1}$

IBC: $$c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}} = w_t + \gamma \frac{w_{t+1}}{1 + r_{t+1}}$$

Next we can set up the Lagrangian and solve.

$$\mathcal{L} = ln(c_{1t}) + \beta ln(c_{2t+1}) + \lambda \left[w_t + \frac{w_{t+1}}{1+r_{t+1}} - c_{1t} - \frac{c_{2t+1}}{1+r_{t+1}} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0: \qquad \lambda = \frac{1}{c_{1t}}$$
$$\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0: \qquad \lambda = \frac{\beta(1+r_{t+1})}{c_{2t+1}}$$

$$\implies \qquad c_{2t+1} = \beta(1+r_{t+1})c_{1t}$$

Plugging this into the budget constraint, we can determine both c_{1t}^*, c_{2t+1}^*, and a_t^*.

$$c_{1t}^* = \frac{1}{1+\beta} \left[w_t + \gamma \frac{w_{t+1}}{1+r_{t+1}} \right]$$

$$c_{2t+1}^* = \frac{\beta(1+r_{t+1})}{1+\beta} \left[w_t + \gamma \frac{w_{t+1}}{1+r_{t+1}} \right]$$

$$a_t^* = \frac{\beta(1+r_{t+1})w_t - \gamma w_{t+1}}{(1+\beta)(1+r_{t+1})}$$

Differentiating a_t^* with respect to γ, we see that assets (and thus savings) are a decreasing function of how much one works in the second period. Intuitively, if you are working more later on in life you don't have to save as much in the first period.

$$\frac{da_t^*}{d\gamma} = -\frac{w_{t+1}}{(1+\beta)(1+r_{t+1})} < 0$$

With regard to the interest sensitivity of savings, we also see that

$$\frac{da_t^*}{dr_{t+1}} = \frac{\gamma w_{t+1}}{(1+\beta)(1+r_{t+1})^2} > 0$$

We see that the interest sensitivity os savings is increasing in γ. If you are in less need of savings because of a high γ, you are more responsive to changes in the interest rate because you are less dependent on savings.

(b) Define the capital-labor ratio k_t as the capital stock divided by *total* labor supply. Explain why $k_{t+1} = a(w_t, w_{t+1}, r_{t+1})/(1+\gamma)$.

Remember that $K_{t+1} = L_t a_t$. We can divide this by L_{t+1} to get the capital-labor ratio.

$$k_{t+1} = \frac{L_t}{L_{t+1}} a_t$$

Also remember that, in this problem, there is no population growth. However, we know that individuals work a fraction γ when old. The young are assumed to supply one unit of labor. That is, in any given period t, we can write the supply of labor next period as $L_{t+1} = (1+\gamma)L_t$. It follows immediately that

$$k_{t+1} = \frac{a_t}{1+\gamma}.$$

(c) Compute the steady state capital stock and the steady state interest rate. Is the aggregate savings rate constant?

First we need to determine the prices. First, the interest rate is going to be given by the marginal product of capital net of depreciation. The wage will be given by labor's share of output.

$$r_t = \alpha k_t^{\alpha-1} - 1 \qquad\qquad w_t = (1-\alpha)k_t^{\alpha}$$

We can use the equation we found in part (b), in conjunction for the expression found for a_t^* in part (a), to get k_{t+1} as a function of k_t.

$$k_{t+1} = \frac{1}{1+\gamma} \frac{\beta(1+r_{t+1})w_t - \gamma w_{t+1}}{(1+\beta)(1+r_{t+1})}$$

$$= \frac{1}{1+\gamma} \frac{\beta\alpha k_{t+1}^{\alpha-1}(1-\alpha)k_t^{\alpha} - \gamma(1-\alpha)k_{t+1}^{\alpha}}{(1+\beta)\alpha k_{t+1}^{\alpha-1}}$$

In the steady state, we'll have that $k_t = k_{t+1} = k^*$. Plugging this in (and simplifying a bit), we have

$$k^* = \left(\frac{\alpha(1-\alpha)\beta}{\alpha(1+\beta)(1+\gamma) + (1-\alpha)\gamma}\right)^{\frac{1}{1-\alpha}}.$$

Finally, we can plug this in to find r^*.

$$r^* = \frac{\alpha(1+\beta)(1+\gamma) + (1-\alpha)(\gamma-\beta)}{(1-\alpha)\beta}$$

The savings rate depends on the interest rate whenever the economy is not in the steady state, and so won't be constant. If the economy is in the steady state, the savings rate must be constant (otherwise the capital stock would be changing from period to period).

Problem 4.3 Consider an overlapping generations economy in which individuals receive exogenous endowment income rather than factor incomes. The emphasis in this problem is on distinguishing the income growth over generations (denoted m) versus income growth over an individual life cycle (denoted g). Individuals of generation t receive an income y_t when young and $(1+g)y_t$ when old. Across generations, endowments grow at the rate m: $y_{t+1} = (1+m)y_t$. Individuals maximize the utility function $U = ln(C_{1t}) + \beta ln(C_{2t+1})$. They can borrow and lend at a constant interest rate r. The population growth rate is a constant n.

(a) How does an increase in the growth rate of individual incomes, g, affect individual savings?

As usual, we begin by finding the intertemporal budget constraint.

First Period: $C_{1t} + a_t = y_t$

Second Period: $C_{2t+1} = (1+r_{t+1})a_t + (1+g)y_t$

IBC: $C_{1t} + \dfrac{C_{2t+1}}{1+r_{t+1}} = \dfrac{2+g+r_{t+1}}{(1+\beta)(1+r_{t+1})}y_t$

Next we can set up the Lagrangian and solve).

$$\mathcal{L} = ln(C_{1t}) + \beta ln(C_{2t+1}) + \lambda\left[\frac{2+g+r_{t+1}}{(1+\beta)(1+r_{t+1})}y_t - C_{1t} - \frac{C_{2t+1}}{1+r_{t+1}}\right]$$

$$\frac{\partial\mathcal{L}}{\partial C_{1t}} = 0: \qquad \lambda = \frac{1}{C_{1t}}$$
$$\frac{\partial\mathcal{L}}{\partial C_{2t+1}} = 0: \qquad \lambda = \frac{\beta(1+r_{t+1})}{C_{2t+1}}$$

$$\implies \qquad C_{2t+1} = \beta(1+r_{t+1})C_{1t}$$

Notice the familiar form of the Euler Equation. Plugging this into the budget constraint, we can determine both C^*_{1t}, C^*_{2t+1}, and a^*_t.

$$C^*_{1t} = \frac{2 + g + r_{t+1}}{(1+\beta)(1+r_{t+1})} y_t$$

$$C^*_{2t+1} = \frac{\beta(2 + g + r_{t+1})}{1+\beta} y_t$$

$$a^*_t = \frac{\beta(1 + r_{t+1}) - (1+g)}{(1+\beta)(1+r_{t+1})} y_t$$

We can see that assets (savings) are negatively related with g. The intuition is that if I expect more income when I get older (higher growth in lifetime income), I will save less today.

$$\frac{da^*_t}{dg} = -\frac{y_t}{(1+\beta)(1+r_{t+1})}$$

(b) How does an increase in m affect the aggregate savings rate, taking g as given? The aggregate savings rate is defined as the ratio of total savings to total endowments.

In each period, the total endowment equals the income of the young plus the income of the old. Let L_t denote the number of young people and y_t denote the income received by young people in period t. Then

$$Y_t = \underbrace{L_t y_t}_{\text{the young}} + \underbrace{\frac{1+g}{(1+m)(1+n)} L_t y_t}_{\text{the old}} = \left[1 + \frac{1+g}{(1+m)(1+n)}\right] L_t y_t.$$

Remember that population grows at a rate n, so the population of old people is $L_t/(1+n)$. Intergenerational income grows at rate m, so we'll similarly have to back out the old people's income by dividing by $1+m$. Last, since old people have income growth of g relative to when they were young, we'll have a factor $1+g$ in there as well.

The next step is to find the aggregate savings in the economy. This is simply given by $L_t a^*_t$, where a^*_t was found in part (a). The aggregate savings rate is thus $S_t = L_t a^*_t / Y_t$.

$$S_t = \frac{\beta(1 + r_{t+1}) - (1+g)}{(1+\beta)(1+r_{t+1})} \Bigg/ \left[1 + \frac{1+g}{(1+m)(1+n)}\right]$$

Notice that an increase in m will decrease the denominator, decreasing the value of S_t because the numerator stays constant. That is, an increase in m leads to an increase in the savings rate holding lifetime income growth constant. This seems to make sense. If the young people have more money with which to save, and the old people's income is growing as it was, the aggregate savings rate (which combines the income of both young and old) must increase.

(c) Assuming $g = m$, how does an equal increase in m and g together affect the aggregate savings rate?

Setting $g = m$, we find that

$$S_t = \frac{\beta(1 + r_{t+1}) - (1 + m)}{(1 + \beta)(1 + r_{t+1})} \Bigg/ \left[1 + \frac{1}{1 + n}\right]$$

Here we see that an increase in m and g by an equal amount will have no effect on the denominator, but serve to decrease the numerator, reducing the aggregate savings rate.

(d) Comment on the following claim: "A country with high income growth tends to have a high savings rate."

The answer is, as per the results in parts (b) and (c), ambiguous. That is, it depends on whether the income growth is coming from inter/intra-generational growth (or, more likely, a combination of the two). If only m is increasing, the additional income accrued to the young generation will cause additional savings to outweigh the dis-savings of the old generation. On the other hand, if both m and g increase equally, we see that these dis-saving incentives are strong, and aggregate savings fall.