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ECON 204A Final Solutions
Fall 2018

This exam is closed book. Most points are given for the correct set-up of a problem and for economically insightful interpretations.

Problem 1 (40p)
This question is about the Ramsey model with productivity growth. Throughout, assume aggregate production is $Y = F(K, AL)$, where F has constant returns to scale and satisfies the Inada conditions, A is productivity, K is capital, and L is population (all working unit time). Productivity A grows at an exogenous rate $g > 0$. Output is used for consumption and capital investment. Population growth n and the depreciation rate δ are exogenous and constant. A representative household maximizes

$$U = \int_0^\infty e^{-\rho t} u(C(t))L(t)dt, \qquad \text{where } u(C) = \frac{C^{1-\theta}}{1-\theta}, \qquad \theta > 0, \theta \neq 1.$$

a) (10p) Set up the Hamiltonian problem with variables K and C. Apply the Maximum Principle, derive the Euler equation, and derive differential equations for k=K/(AL) and c = C/A.

$$\mathcal{H}(C, K, \lambda, t) = e^{-\rho t} u(C)L + \lambda \left[F(K, AL) - \delta K - CL\right] \tag{1}$$

$$\frac{\partial \mathcal{H}}{\partial C} = 0 : \qquad \lambda L = e^{-\rho t} C^{-\theta} L \tag{2}$$

$$\frac{\partial \mathcal{H}}{\partial K} = -\dot{\lambda} : \quad \frac{\dot{\lambda}}{\lambda} = -[F_K(K, AL) - \delta] = -r \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{K} : \quad \dot{K} = F(K, AL) - \delta K - CL \tag{4}$$

Taking logs and substituting c into (2) gives us:

$$ln(\lambda) = -\rho t - \theta(ln(c) + ln(A))$$

Taking time derivatives:

$$\frac{\dot{\lambda}}{\lambda} = -\rho - \theta(\frac{\dot{c}}{c} + g)$$

Equating this with (3) and solving for $\dot{c}/c$, we get the Euler Equation/differential equation for c:

$$\frac{\dot{c}}{c} = \frac{r - \rho - \theta g}{\theta} \tag{5}$$

Now, turning to our differential equation for k:

$$k = \frac{K}{AL}$$

Taking logs and time derivatives:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}$$

Plugging in (4)

$$\frac{\dot{k}}{k} = \frac{F(K,AL) - \delta K - CL}{K} - g - n \Longrightarrow \dot{k} = f(k) - c - (\delta + g + n)k$$

Thus, we have:

$$\frac{\dot{c}}{c} = \frac{r - \rho - \theta g}{\theta} \tag{5}$$

and

$$\dot{k} = f(k) - c - (\delta + g + n)k \tag{6}$$

b) (5p) Express U in terms of c and set up the Hamiltonian problem with variables k and c. Show that applying the Maximum Principle yields the same differential equations as in (a).

Building off our previous derivation of $\dot{k}$, we can set up our Hamiltonian:

$$\mathcal{H}(c,k,\lambda,t) = e^{-\rho t}u(cA)L + \lambda\left[f(k) - c - (\delta + g + n)k\right] \tag{7}$$

$$\frac{\partial \mathcal{H}}{\partial c} = 0 : \qquad \lambda = e^{-\rho t}AL(cA)^{-\theta} \tag{8}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} : \quad \frac{\dot{\lambda}}{\lambda} = -[f'(k) - (\delta + g + n)] = -[r - (g + n)] \tag{9}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{k} : \qquad \dot{k} = f(k) - c - (\delta + g + n)k \tag{10}$$

We can see already that (10) and (6) are the same. Turning to (8), taking logs and time derivatives gives us:

$$\frac{\dot{\lambda}}{\lambda} = -\rho + n + g - \theta(\frac{\dot{c}}{c} + g)$$

Equating this with (9) and solving for $\dot{c}/c$, we get the Euler Equation/differential equation for c:

$$\frac{\dot{c}}{c} = \frac{r - \rho - \theta g}{\theta} \tag{11}$$

Which we can see is the same as (5).

c) (10p) Set up the Phase diagram: define the steady state conditions and the equations for constant c and constant k. Explain the directions of the Phase arrows in the various regions, explain the saddle path, and explain why the economy converges to (k^*, c^*) from any $k > 0$. [Note: You may assert without proof that some paths satisfy/violate transversality conditions.]

For a constant c and k we will set $\dot{c}$ and $\dot{k}$ equal to zero, giving us our conditions.

For $\dot{c} = 0$:

$$f'(k^*) = \delta + \rho + \theta g$$

and for $\dot{k} = 0$

$$c = f(k) - (\delta + n + g)k$$

These intersection of these two curves pins down c^* and k^* while the curves themselves help us describe our phase diagram. Turning first to our $\dot{c} = 0$ locus, we can see that due to the concavity of the production function, if k is low f'(k) is high and $\dot{c} > 0$. Likewise, if k is high then we know f'(k) is low and $\dot{c} < 0$. So if we are to the left of the $\dot{c} = 0$ curve then consumption is rising, whereas to the right consumption is falling.

Now, looking at our $\dot{k} = 0$ equation, we can see that for a given level of k if c is high then $\dot{k} < 0$ whereas if c is low then $\dot{k} > 0$. Thus if we are below the $\dot{k} = 0$ locus than capital is increasing, whereas if we are above it then capital is declining. These forces make up the arrows on the phase diagram. The transversality condition and the No-ponzi condition, in conjunction with optimizing behavior, guarantee convergence as they preclude paths in the upper left or bottom right quadrants of the diagram. For a given level of k, agents will choose their consumption such that they land on the saddle path in either the bottom left or top right quadrants.

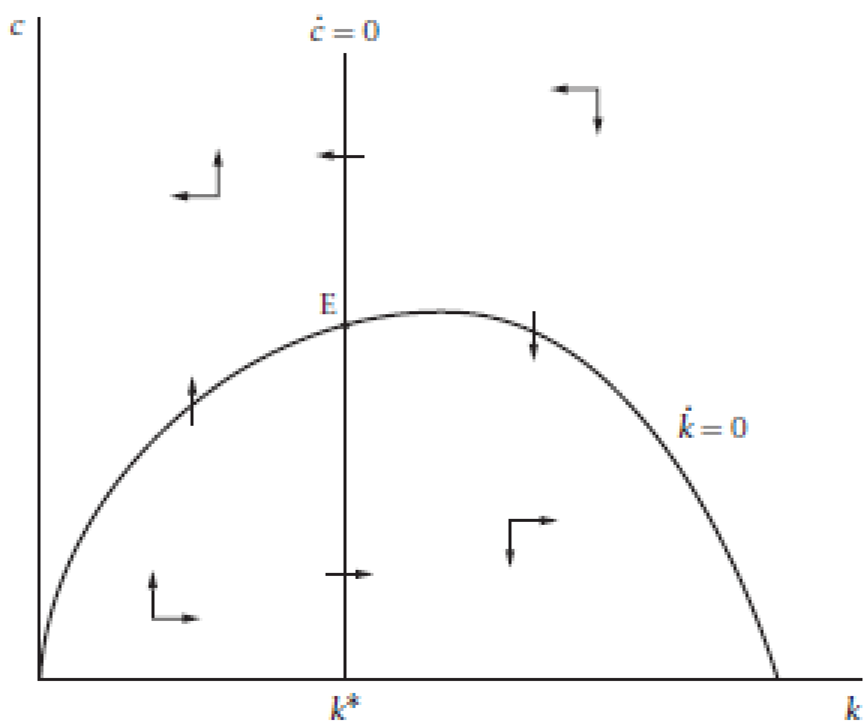
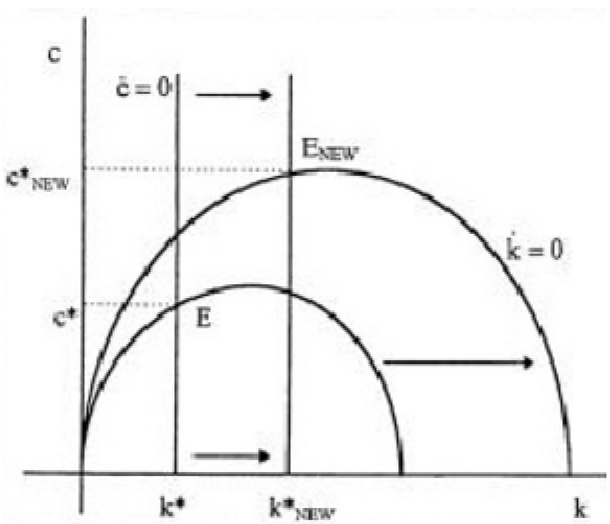


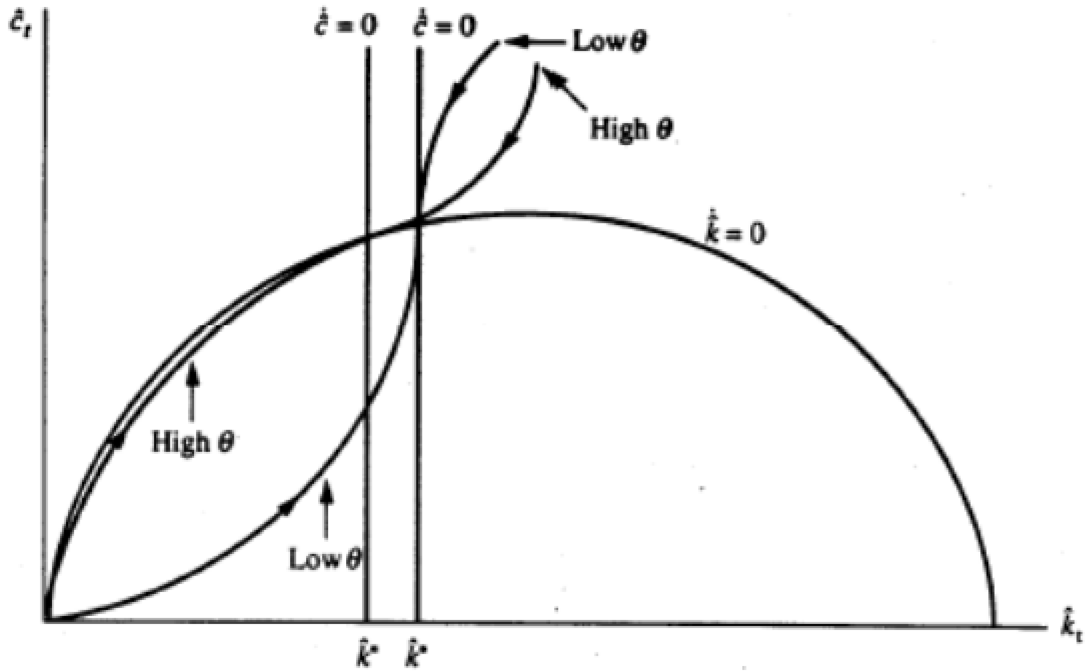
FIGURE 2.3 The dynamics of c and k

d) (10p) Suppose the economy is in steady state with $g = g_0$ at time $t = 0$. Unexpectedly, productivity growth declines to $g = g_1 < g_0$ for all $t0$. Describe the economic impact, on the steady state and during the transition. Explain why $c(0)$ may jump up or down, depending on certain parameters. Illustrate the two possibilities with two Phase diagrams, one assuming θ is near zero, the other assuming $\frac{1}{\theta}$ is near zero.

First, looking at our $\dot{c} = 0$ equation we can see that an decrease in g will require a decline in $f'(k^*)$. Thus, due to the concavity of $f(\cdot)$, we know that the $\dot{c} = 0$ curve will shift to the right. Looking at our $\dot{k} = 0$ equation, we can see that a decrease in g will allow for higher levels of consumption for each level of k. Thus the $\dot{k} = 0$ curve will shift upwards. The following figure illustrates these shifts:



It is clear that the new steady state will have both a higher c^* and and higher k^*. However, we cannot speak as to what happens immediately after the change without knowing what our θ is. To see this note the following figure:



For a high level of θ (thus when $\frac{1}{\theta}$ is near zero, we can see that the curve closely saddles up to the $\dot{k}=0$ curve. In this scenario, when the shift occurs we can see clearly that the new saddle path will be below the old one for all levels of k, whereas the opposite will hold true when θ is near zero.

Problem 2 (25p)

Suppose aggregate production $Y = F(K, AL)$ has constant returns to scale and satisfies the Inada conditions. Capital K depreciates at rate δ. Productivity and population L are constant. Households have preferences over per-capita private consumption C and per-capita government-provided goods G:

$$U = \int_0^\infty e^{-\rho t} u(C(t), G(t)) L(t) dt, \quad \text{where } u(C,G) = ln(C) + \gamma(t) \cdot ln(G), \quad \gamma > 0$$

The weight on public goods $\gamma(t) > 0$ is exogenous and may vary over time. The government pays for G with lump-sum taxes $T = G$.

a) (10p) Suppose a social planner maximizes U by choice of (C,G) and assume $\gamma(t) = \gamma$ is constant. Set up the Hamiltonian problem, apply the Maximum Principle, and derive differential equations for C and K. [Hint: you can eliminate G.]

First we will set up our problem with both choice variables (while normalizing L to 1):

$$\mathcal{H}(C, G, K, \lambda, t) = e^{-\rho t}(ln(C) + \gamma(t)ln(G)) + \lambda [F(K, 1) - C - \delta K - G] \tag{7}$$

$$\frac{\partial \mathcal{H}}{\partial C} = 0: \qquad \lambda = e^{-\rho t} \cdot \frac{1}{C} \tag{1}$$

$$\frac{\partial \mathcal{H}}{\partial G} = 0: \qquad \lambda = e^{-\rho t} \cdot \frac{\gamma(t)}{G} \tag{2}$$

$$\frac{\partial \mathcal{H}}{\partial K} = -\dot{\lambda}: \quad \frac{\dot{\lambda}}{\lambda} = -[F_K(K, 1) - \delta] = -r \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{K}: \quad \dot{K} = [F(K, 1) - C - \delta K - G] \tag{4}$$

Now, note that combining (1) and (2) gives us:

$$G = \gamma(t)C$$

Thus, we can replace G with γC moving forward. Following the same steps as in problem 1, we get our standard Euler Equation:

$$\frac{\dot{C}}{C} = \frac{F_K(K, 1) - \delta - \rho}{\theta}$$

and, after substituting in for G, we get:

$$\dot{K} = F(K, 1) - \delta K - (1 + \gamma)C$$

b) (15p) Suppose the economy is in steady state with $\gamma(t) = \gamma_0$. At time $t = 0$, the government discovers a better way to deliver public goods, but the reforms will take time. Specifically, assume $\gamma(t) = \gamma_0$ stays constant until time $t_1 > 0$. For $t \geq t_1$, the weight $\gamma(t) = \gamma_1 > \gamma_0$ will be higher than before. Describe how the social planner should respond to this discovery. Illustrate your answer in a Phase diagram. Sketch the time series for C, G, and K.

First, note that the discovery will note have an effect on the $\dot{C} = 0$ curve. It will, however, cause the $\dot{K} = 0$ to shift down. Thus we know that our new steady state will have the same (k^*) with a lower (c^*). Now, note that because the change is anticipated households will smooth their consumption habits to adapt to the change in time t = 0. The following describes (in words) what the time paths look like for each variable:

C: At time t=0 agents will lower their consumption and begin accumulating capital in anticipation of the new saddle path. Thus there will be a discrete drop in c, followed by a gradual decline until it reaches the new equilibrium.

K: As mentioned before, agents will begin accumulating capital in anticipation of the change. At time t_1, when the change occurs, the economy will be on the new saddle path. Thus capital will begin declining, eventually returning to the pre-change level.

G: The path of G is a bit more complicated. Between t_0 and t_1 it will follow the same path as consumption: A discrete drop followed by a gradual decline while the agents accumulate capital. At time t_1, however, G will discretely jump up when the reforms take effect. It will then gradually decline until the economy reaches the new steady state.

Problem 3 (35p)

Consider the following overlapping generations economy. Individuals in generation t maximize a logarithmic utility function $U = ln(C_{1t}) + \beta ln(C_{2t+1})$ over working-age and retirement-age consumption, where $0 < \beta < 1$. Technology is Cobb-Douglas with capital share α and 100% depreciation. Productivity is constant. Unless noted otherwise, $L_{t+1} = (1+n)L_t$ grows at rate $n > 0$. The wage W_t and the interest rate r_{t+1} are determined competitively.

The government operates a social security system such that transfers $TR_t = \rho \cdot W_t$ are a constant fraction $\rho \in (0,1)$ of the wages earned by the young (the so-called replacement rate). The system is pay-as-you go, so $T_{1t} = \frac{TR_t}{(1+n)}$.

a) (10p) Set up the individual optimization problem for generation t. Solve for optimal asset holdings as function of W_t, r_{t+1}, and ρ .

First we should construct our IBC:

First Period: $c_{1t} + a_t = w_t - T_t$

Second Period: $c_{2t+1} = (1 + r_{t+1})a_t + TR_{t+1}$

IBC: $c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - T_t + \frac{TR_{t+1}}{1+r_{t+1}}$

Noting that $T_t = \frac{\rho W_t}{(1+n)}$ and $TR_{t+1} = \rho W_{t+1}$. The Lagrangian and associated FOCs look very similar to what we've seen previously.

$$\mathcal{L} = ln(c_{1t}) + \beta ln(c_{2t+1}) + \lambda \left[w_t - T_t + \frac{TR_{t+1}}{1+r_{t+1}} - c_{1t} - \frac{c_{2t+1}}{1+r_{t+1}} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0: \qquad \lambda = \frac{1}{c_{1t}}$$
$$\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0: \qquad \lambda = \frac{\beta(1+r_{t+1})}{c_{2t+1}}$$

$$\implies \qquad c_{2t+1} = \beta(1+r_{t+1})c_{1t}$$

Plug the EE back into the IBC to obtain an expression for c_{1t} and, then, a_t. Beginning with c_{1t} we get:

$$c_{1t} = \frac{w_t}{1+\beta} - \frac{T_{1t}}{1+\beta} + \frac{TR_{t+1}}{(1+r_{t+1})(1+\beta)}$$

Now plugging this into our first period budget constraint (and substituting in for T_{1t} and TR_{t+1}:

$$a_t = \frac{\beta}{1+\beta} W_t \left(1 - \frac{\rho}{1+n}\right) - \frac{1}{1+\beta} \frac{\rho W_{t+1}}{1+r_{t+1}} \tag{1}$$

b) (10p) Derive the mapping from k_t to k_{t+1} for this economy and show there is a unique steady state k^*. Show how k^* depends on ρ and on n. Show that the economy is dynamically efficient if α is high enough.

Recall that $K_{t+1} = L_t a_t$. As productivity is constant, dividing by L_{t+1} will give us:

$$k_{t+1} = \frac{1}{1+n} a_t$$

Plugging in (1):

$$k_{t+1} = \frac{1}{1+n}\left(\frac{\beta}{1+\beta} W_t \left(1 - \frac{\rho}{1+n}\right) - \frac{1}{1+\beta}\frac{\rho W_{t+1}}{1+r_{t+1}}\right) \tag{2}$$

Now, recall that with Cobb-Douglas production and 100% depreciation:

$$W_t = (1-\alpha)k_t^{\alpha}$$

and

$$1 + r_t = \alpha k_t^{\alpha-1}$$

Plugging these into (2):

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n}\right)(1-\alpha)k_t^{\alpha} - \frac{\rho}{1+n}\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}k_{t+1}$$

and solving for k_{t+1}:

$$k_{t+1} = \frac{\frac{1}{1+n}\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n}\right)(1-\alpha)}{1 + \frac{\rho}{1+n}\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}} k_t^{\alpha}$$

Now, recall that a unique steady state exists if $\frac{dln(k_{t+1})}{dln(k_t)} < 1$. Note that:

$$\frac{dln(k_{t+1})}{dln(k_t)} = \alpha < 1$$

So we know a unique steady state exists. Thus, in the steady state:

$$k^* = \left(\frac{\frac{1}{1+n}\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n}\right)(1-\alpha)}{1 + \frac{\rho}{1+n}\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}}\right)^{\frac{1}{1-\alpha}}$$

Now, for simplicity let's define:

$$\omega = \frac{\frac{1}{1+n}\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n}\right)(1-\alpha)}{1 + \frac{\rho}{1+n}\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}} = \frac{\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n}\right)(1-\alpha)}{1 + n + \rho\frac{1}{1+\beta}\frac{1-\alpha}{\alpha}}$$

Thus we have:

$$k^* = \omega^{\frac{1}{1-\alpha}}$$

Note first that the numerator of ω is decreasing in ρ while the denominator is increasing in ρ. This tells us that $\frac{\partial k^*}{\partial \rho} < 0$. Seeing the effect of n is a bit trickier here. Taking a derivative:

$$\frac{\partial \omega}{\partial n} = -\frac{\omega - \frac{\beta}{1+\beta}(1-\alpha)\frac{\rho}{(1+n)^2}}{1 + n + \rho\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}}$$

Which is positive for large ρ and negative for small ρ. Finally, note that the economy is dynamically efficient if $r^* > n$. Plugging our k^* into the r equation:

$$1 + r^* = \alpha \left(\frac{\frac{1}{1+n}\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n}\right)(1-\alpha)}{1 + \frac{\rho}{1+n}\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}} \right)^{\frac{\alpha-1}{1-\alpha}}$$

Which simplifies to:

$$1 + r^* = \frac{(1+n)^2(1+\beta)\alpha + \rho(1-\alpha)(1+n)}{\beta(1+n-\rho)(1-\alpha)}$$

Thus we have dynamic efficiency when:

$$\frac{(1+n)^2(1+\beta)\alpha + \rho(1-\alpha)(1+n)}{\beta(1+n-\rho)(1-\alpha)} > (1+n)$$

Or:

$$\frac{(1+n)(1+\beta)\alpha + \rho(1-\alpha)}{\beta(1+n-\rho)(1-\alpha)} > 1$$

Which simplifies further to:

$$\frac{\alpha}{1-\alpha} > \frac{\beta(1+n-\rho) - \rho}{(1+n)(1+\beta)}$$

Which will hold true for a sufficiently high α (Think about what happens to the left side of the equation as α approaches 1).

c) (10p) Starting in steady state with $n > 0$, suppose population growth stops in period t=1. That is, $L_t = L_1 = (1+n)L_0$ for all $t > 1$. Explain how the economy adjusts.

First, note that when $n = 0$ we do not know if the new steady state level of capital will be higher or lower than the previous one. As we showed in part b), this depends on the value of ρ. We can, however, speak as to what occurs during the transition. Note that we have:

$$k_{t+1} = \frac{\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n_t}\right)(1-\alpha)}{1 + n_{t+1} + \rho\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}} k_t^{\alpha}$$

Now, we can break down the effect by generations. It is clear that the generation born in $t = 0$ will be unaffected by the change. The first generation to be affected will be those born in t=1. While young the taxes they pay will be the same, but we can see by the above transition equation that k_2 will be higher (Note that $k_{t+1} = a_t$). For every generation beginning in t = 2, however, the young will be paying higher taxes relative to the previous generation. k_t will begin to fall, eventually converging to the new steady state.

d) (10p) Let the model be modified to include a childhood period. Children are not economically active, but working-age adults (parents) must spend an amount $\chi \cdot W_t$ per child to raise them. Assume $0 < \chi < \frac{1}{2+n}$ is small enough to be feasible. The working-age budget equation is then $a_t + C_{1t} + (1+n) \cdot \chi \cdot W_t = W_t T_{1t}$. The model is otherwise unchanged. Explain to what extent your answers in (a)-(c) are different in the modified model.

For simplicity let's break down the changes by their respective sections:

a) In part a, first and foremost, we must change our period-1 budget constraint to:

$$a_t + C_{1t} = W_t \left(1 - \frac{\rho}{1+n} - (1+n)\chi\right)$$

which will, in turn, change our a_t equation to:

$$a_t = \frac{\beta}{1+\beta} W_t \left(1 - \frac{\rho}{1+n} - (1+n)\chi\right) - \frac{1}{1+\beta}\frac{\rho W_{t+1}}{1+r_{t+1}}$$

b) In part b, we will see a change in our transition function (and consequently our steady state value of capital). The transition function becomes:

$$k_{t+1} = \frac{\frac{\beta}{1+\beta}\left(1 - \frac{\rho}{1+n_t} - (1+n_{t+1})\chi\right)(1-\alpha)}{1 + n_{t+1} + \frac{\rho}{1+n}\frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}} k_t^{\alpha}$$

we can see that relative to our previous finding in b), in this economy there will be less savings, a lower k^*, and this a higher r^*. Thus, there is a greater likelihood of being dynamically efficient.

c) In part c, we will see that when $n = 0$ the cost of childcare is reduced. The basic movement of the economy will be identical, but k_2 will increase even more relative to k_1 and the steady state value of k will be different.