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Econ 204A
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Final Exam
Fall 2016

This exam is closed book. Most points are given for the correct set-up of a problem and for economically insightful interpretations. You have 3 hours for a maximum score of 100 points.

[30] **Problem 1.** Consider a continuous-time economy with constant population, constant productivity, and depreciation rate $\delta > 0$. Households maximize

$$\int_0^\infty e^{-\rho t} u\left(c(t)\right) dt$$

subject to the capital accumulation equation $\dot{k} = f(k) - c - G - \delta k$, where k is the capital stock, c is consumption, and $G = G(t)$ is an exogenous path of government expenditures; $u(c)$ and $f(k)$ are increasing and concave. Spending is financed by lump-sum taxes.

[10] **(a)** State the Hamiltonian problem, apply the maximum principle, and derive equations that describe the dynamics of consumption and capital. Explain the intuition for these equations.

$$\mathcal{H}(c, k, \lambda, t) = e^{-\rho t} u(c) + \lambda \left[f(k) - c - G - \delta k\right] \tag{1}$$

$$\frac{\partial \mathcal{H}}{\partial c} = 0 : \qquad \lambda = e^{-\rho t} u'(c) \tag{2}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} : \quad \frac{\dot{\lambda}}{\lambda} = -[f'(k) - \delta] \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{k} : \qquad \dot{k} = f(k) - c - G - \delta k \tag{4}$$

Using (2), take logs and differentiate w.r.t. t.

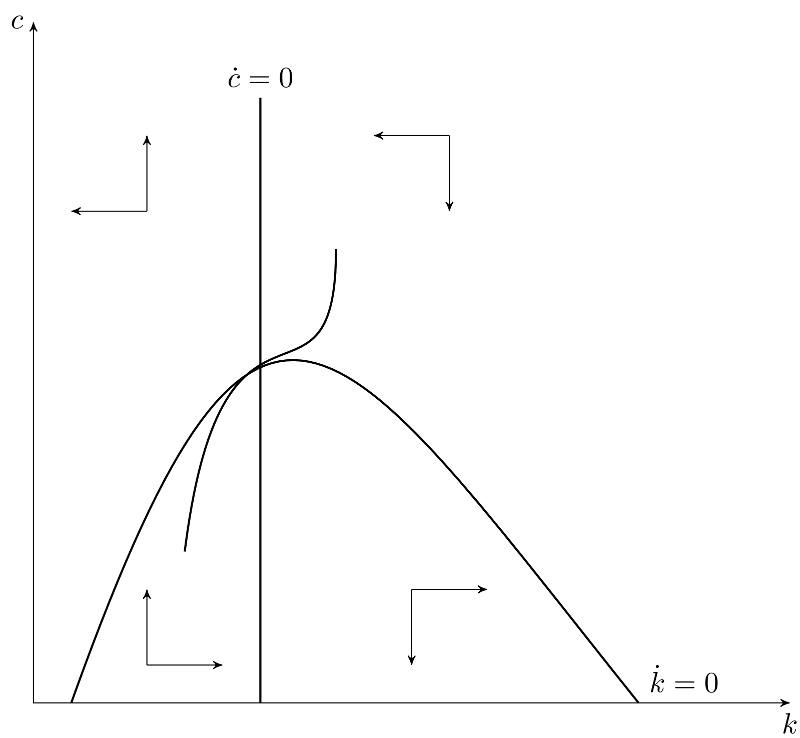
$$\frac{\dot{\lambda}}{\lambda} = -\rho + \frac{u''(c)c}{u'(c)} \frac{\dot{c}}{c} = -\rho - \theta \frac{\dot{c}}{c}$$

Equate this result with (3) to get the EE.

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta} \tag{5}$$

The dynamics of k and c are given by (4) and (5), respectively. With regard to intuition, the dynamics of k are a fairly mechanical "accounting" of the uses of output. Whatever is not consumed, spent by the government, or depreciated will contribute to the change in capital. With regard to the dynamics of c, we can see that consumption growth is positively related with the interest rate and the elasticity of intertemporal substitution (EIS). A higher r enables more future consumption. Further, a high EIS (i.e. $1/\theta$) means that the agent is more responsive to this interest rate, substituting consumption today with more consumption tomorrow. On the other hand, it is negatively related with the rate of time preference: if an individual is more impatient, she will prefer to consume more today relative to tomorrow.

[5] **(b)** Construct the phase diagram for this problem. Assuming $G(t) = G_0$ is constant for all t. Explain briefly why capital and consumption converge to (k^*, c^*) from any initial $k(0) > 0$.



With regard to convergence, briefly consider both the $\dot{c} = 0$ and $\dot{k} = 0$ curves.

$$\underbrace{f'(k^*) = \delta + \rho}_{\dot{c}=0} \qquad\qquad \underbrace{c = f(k) - G - \delta k}_{\dot{k}=0}$$

First, note that for any $k < k^*$ (but still positive), we can see that $\dot{c} > 0$ (recall f is concave). I.e., c is too low (for an optimizing agent). Looking over at the $\dot{k} = 0$ curve, if c is growing the RHS must also be growing. Because of the Inada conditions we know that $f(k) > \delta k$ for low levels of k. We can then determine that $\dot{k} > 0$. A mirrored argument establishes the opposite for $k > k^*$. Putting everything together, for any positive $k(0)$, we'll have the economy converging toward a steady state given by (k^*, c^*).

[15] **(c)** Suppose it becomes known at time $t = 0$ that there will be a burst of government spending over the time interval $[t_1, t_2]$, where $0 < t_1 < t_2$. During this time interval, $G(t) = G_1 > G_0$. Assume $k(0) = k^*$. Explain the impact of this change in government spending on consumption, on capital, and on interest rates over time. Illustrate your answers in a phase diagram and with time series charts for $c(t)$, $k(t)$, and $r(t)$.

The policy will shift the $\dot{k} = 0$ curve down. Because this is change is anticipated, optimizing agents will (at the time of the announcement) discretely lower their consumption a small amount. The dynamics will then cause k to increase and c to decrease until t_1, at which point the new dynamics governing the economy will cause k to start falling (c will remain decreasing). Eventually the $\dot{c} = 0$ curve will be crossed and c will begin to rise (k remains decreasing). This final stretch will, at t_2, place the economy back on the original saddle path, where convergence to the steady state will ensue.

[30] **Problem 2.** Consider a continuous-time economy. A representative household maximizes

$$\int_0^\infty e^{-\rho t} u\left(C(t)\right) L(t) dt$$

where $\rho > 0$ and where $u(C)$ is power utility with elasticity of intertemporal substitution $1/\theta$. Output is $Y = K^\alpha L^{1-\alpha}$ and satisfies the standard assumptions. Labor L grows at rate $n > 0$. Total factor productivity is constant except as noted below. Capital depreciates at a constant rate $\delta > 0$.

[10] **(a)** Set up the Hamiltonian problem, apply the maximum principle, and derive differential equations for optimal C and $k = K/L$.

$$\mathcal{H}(C, k, \lambda, t) = e^{-\rho t} u(C) L + \lambda \left[f(k) - C - (n + \delta)k\right] \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial C} = 0 : \qquad \lambda = e^{-\rho t} u'(c) L \tag{7}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} : \quad \frac{\dot{\lambda}}{\lambda} = -[f'(k) - (n + \delta)] \tag{8}$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{k} : \qquad \dot{k} = f(k) - C - (n + \delta)k \tag{9}$$

Take logs and then a time derivative of (7).

$$\frac{\dot{\lambda}}{\lambda} = -\rho + \frac{u''(C)C}{u'(C)}\frac{\dot{C}}{C} + n = -\rho - \theta\frac{\dot{C}}{C} + n$$

Equate the last result with (8) to get the EE.

$$\frac{\dot{C}}{C} = \frac{f'(k) - \delta - \rho}{\theta} \tag{10}$$

The dynamics for k and C are given by (9) and (10), respectively. Plugging in for the Cobb-Douglas functional form we have the following.

$$\dot{k} = k^{\alpha} - C - (n+\delta)k \tag{9'}$$

$$\frac{\dot{C}}{C} = \frac{\alpha k^{\alpha-1} - \delta - \rho}{\theta} \tag{10'}$$

[5] **(b)** Show that the economy converges to a unique steady state (C^*, k^*) from any positive initial value $K(0) > 0$.

See earlier answer from Problem 1.

[10] **(c)** Suppose an unexpected technical innovation at time $t = t_0$ increases output by a factor of $\varphi > 1$, so $Y = \varphi K^{\alpha} L^{1-\alpha}$ for all $t > t_0$. Assume the economy starts in the steady state. Show how this innovation changes the time paths of consumption and capital. Explain how the responses of consumption and capital depend on the elasticity of intertemporal substitution. (Hint: draw separate diagrams for cases with high and low values of $1/\theta$.)

To get started, let's look at the $\dot{C} = 0$ and $\dot{k} = 0$ curves for this problem. Using the functional form, note that the $\dot{C} = 0$ curve pins down k^*.

$$\underbrace{k^* = \left(\frac{\alpha\varphi}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}}_{\dot{C}=0} \qquad \underbrace{C = \varphi k^{\alpha} - (n+\delta)k}_{\dot{k}=0}$$

We can see that the technological innovation will shift the $\dot{C} = 0$ curve to the right ($\varphi > 1$) and it will move the $\dot{k} = 0$ curve upward. In general, because both curves are shifting this will lead to some ambiguities as far as what exactly happens to k and C over time (we know eventually that the steady state levels of each variable will be higher).

First, for low values of of $1/\theta$, the saddle paths will be close to the $\dot{k} = 0$ curve. Assuming that the EIS is sufficiently low, we can figure that the new saddle path will be above the old saddle path. In this scenario, optimizing agents will increase their consumption discretely at the time of the change. From here, both c and k increase to the new, higher steady state. On the other hand, for high EIS, the saddle path will "hug" the $\dot{c} = 0$ curve. If the EIS is sufficiently high, the new saddle path will be below, and consumption will discretely decrease at the time of the change. As before, capital and consumption will then increase to the new, higher steady state.

[5] **(d)** How would your answer in (c) change if the innovation is discovered at time $t = t_0$ but does not become productive until a later time $t = t_1 > t_0$?

First consider the low EIS situation. If the innovation is anticipated, optimizing agents will discretely increase their consumption, but not as much as with the unanticipated case. A major difference here, though, is that capital will temporarily decline until t_1, when the economy will have been optimally moving toward, and land on, the new saddle path. From here c and k will increase to the new steady state.

Next, for high EIS, agents will slightly decline (discretely) their consumption. c will continue to decline while k increases until t_1, at which point the economy will have been placed on the new saddle path, and c and k will increase to the new steady state.

[40] **Problem 3.** Consider an overlapping generations economy. Production is Cobb-Douglas with capital share $\alpha \in (0, 1)$ and 100% depreciation. There is no population growth ($L = 1$) and no productivity growth ($A = 1$). The government spends an exogenous share $\gamma \in (0, 1)$ of output, $G_t = \gamma Y_t$. Individuals maximize $U = u(C_{1t}) + \beta u(C_{2t+1})$ and only work in the first period of their life, earning W_t.

For parts (a)-(c), assume utility is logarithmic, $u(C) = ln(C)$.

[10] **(a)** Suppose G is financed by a proportional tax τ_{W_t} on wage income.

i. Explain why $\tau_{W_t} = \frac{\gamma}{1-\alpha}$ for all t if the government's budget is balanced.

We are told that government spending is financed by this proportional tax on wage income. First note that, with $g = n = 0$, we can quickly translate things into effective units (by taking your finger off of the "shift" key). In any regard, we have the following.

$$
\begin{aligned}
G_t &= \tau_{W_t} W_t \\
\gamma Y_t &= \tau_{W_t} w_t \\
\gamma f(k_t) &= \tau_{W_t}[f(k_t) - k_t f'(k_t)] \\
\gamma k_t^{\alpha} &= \tau_{W_t}[(1-\alpha)k_t^{\alpha}]
\end{aligned}
$$

$$
\implies \qquad \tau_{W_t} = \frac{\gamma}{1-\alpha} \tag{11}
$$

In words, in order for the government to pay for a fraction γ of output by only taxing wages, it will need to scale up the tax rate to account for the fact that wages reflect labor's share of income $(1-\alpha)$.

ii. Determine k^* and explain why k^* is decreasing in γ.

First let's derive the IBC.

First Period: $C_{1t} + a_t = (1 - \tau_{W_t})w_t$

Second Period: $C_{2t+1} = (1 + r_{t+1})a_t$

IBC: $$c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}} = (1 - \tau_{W_t})w_t \tag{12}$$

The Lagrangian, FOCs, and EE are then

$$
\mathcal{L} = ln(c_{1t}) + \beta ln(c_{2t+1}) + \lambda\left[(1-\tau_{W_t})w_t - c_{1t} - \frac{c_{2t+1}}{1+r_{t+1}}\right]
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0: &\qquad \lambda = \frac{1}{c_{1t}} \\
\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0: &\qquad \lambda = \frac{\beta(1+r_{t+1})}{c_{2t+1}}
\end{aligned}
$$

$$
\implies \qquad c_{2t+1} = \beta(1 + r_{t+1})c_{1t}
$$

Skipping ahead to savings, a_t, we can determine that

$$
a_t = \frac{\beta}{1+\beta}(1 - \tau_{W_t})w_t. \tag{13}
$$

Because the government spending is paid for by the tax revenues, we know that assets aggregate as follows.

$$K_{t+1} = L_t a_t \qquad \Longrightarrow \qquad k_{t+1} = a_t$$

We can also plug in for $\tau_{W_t} = \gamma/(1-\alpha)$ and $w_t = (1-\alpha)k_t^\alpha$.

$$k_{t+1} = \frac{\beta(1-\alpha-\gamma)}{1+\beta} k_t^\alpha$$

Next impose the steady state condition, $k_t = k_{t+1}$, to obtain k^*.

$$k^* = \left(\frac{\beta(1-\alpha-\gamma)}{1+\beta}\right)^{\frac{1}{1-\alpha}} \tag{14}$$

We can see that k^* is decreasing in γ. This makes sense because, in this model, the government is "burning money" insofar as the more it spends (which is proportional to γ) the less is available to invest in capital.

[10] **(b)** Suppose the government considers introducing a tax on firms to reduce the wage tax. The tax on firms is levied at rate τ_{F_t} on sales, the difference between output and wages, so revenues are $\tau_{F_t}(Y_t - W_t)$. The return on savings is reduced accordingly, so $r_t = (1-\tau_{F_t})f'(k_t) - 1$.

i. Explain why government budget balance requires $\gamma = \alpha\tau_{Ft} + (1-\alpha)\tau_{Wt}$ for all t.

$$\begin{aligned} G_t &= \tau_{F_t}(Y_t - W_t) + \tau_{W_t} W_t \\ \gamma f(k_t) &= \tau_{F_t}(f(k_t) - (1-\alpha)f(k_t)) + \tau_{W_t}(1-\alpha)f(k_t) \\ \gamma f(k_t) &= \tau_{F_t}\alpha f(k_t) + \tau_{W_t}(1-\alpha)f(k_t) \end{aligned}$$

$$\Longrightarrow \qquad \gamma = \alpha\tau_{F_t} + (1-\alpha)\tau_{W_t} \tag{15}$$

In words, in order to pay for the fraction γ of output, the government must tax each factor's share of output accordingly to maintain a balanced budget.

ii. Show that when (τ_F, τ_W) are constant, k^* depends on τ_W but not on τ_F (or, equivalently, show k^* that depends on $\gamma - \alpha\tau_F$). Explain why this makes sense economically.

Notice that the problem for a young agent from part (a) is essentially unchanged (the interest rate will invariably affected by the tax on firms). Because of the log utility specification, the r_{t+1} will end up canceling out of the expression for assets. Using equation (13), we can plug in for τ_W from (15).

$$a_t = \frac{\beta}{1+\beta}\left(1 - \frac{\gamma - \alpha\tau_F}{1-\alpha}\right)(1-\alpha)f(k_t)$$

Plugging in for $f(\cdot)$, aggregating, and simplifying, we find that

$$k_{t+1} = \frac{\beta}{1+\beta}(1 - \alpha - \underbrace{(\gamma - \alpha\tau_F)}_{=(1-\alpha)\tau_W})k_t^{\alpha}$$

$$= \frac{\beta(1-\alpha)(1-\tau_W)}{1+\beta}k_t^{\alpha}.$$

We can see that the expression does not depend on τ_F, but for completeness the steady state value of k is given by

$$k^* = \left(\frac{\beta(1-\alpha)(1-\tau_W)}{1+\beta}\right)^{\frac{1}{1-\alpha}}. \tag{16}$$

Economically this makes sense because capital accumulation depends on individual savings decisions. With log utility, the savings rate is constant and doesn't depend on r_{t+1}, meaning that it won't depend on τ_F.

[10] **(c)** Suppose the economy enters period $t = 1$ in a steady state with given $\gamma > 0$ and $\tau_F = 0$. In $t = 1$, the government unexpectedly announces that $\tau_{Ft} = \gamma$ for all $t \geq 1$, where γ remains unchanged. Also assume that the government always maintains a balanced budget. Describe and explain the transition process to the new steady state. (Hint: use graphs to document your answers.)

If $\tau_F = \gamma$, we must also have that $\tau_W = \gamma$ (see (15)). We also know that, when there is no tax on firms, that the original steady state will be given by (14). After this change, the steady state will be given by

$$k^{*'} = \left(\frac{\beta(1-\alpha)(1-\gamma)}{1+\beta}\right)^{\frac{1}{1-\alpha}}. \tag{17}$$

You can easily show that $k^{*'} > k^*$. The intuition is that, by also taxing firms, the burden on savers is reduced, which induces more savings (and thus a larger capital stock). After the change, the **k** function will rotate up, and capital will start to increase until the new steady state is reached.

[10] **(d)** Explain how the results in (a)-(c) would differ under power utility, $u(C) = \frac{1}{1-\theta}C^{1-\theta}$. For example, do any of the taxes have qualitatively new effects? Does a higher / lower elasticity of intertemporal substitution reduce or magnify the impact of tax changes? (Hint: graphical arguments and economic logic suffice.)

With power utility, the savings rate will depend (positively) on the interest rate. Recall that the EE for power utility is $c_{2t+1} = (\beta(1 + r_{t+1}))^{\frac{1}{\theta}} c_{1t}$. Furthermore, savings will be more sensitive to changes in the interest rate for higher EIS ($\Longleftrightarrow$ lower θ). Because the interest rate is affected by the tax on firms, τ_F, k^* will depend on both taxes. Because assets (and thus capital) will depend positively on the interest rate, taxes on firms will thus be magnified for low EIS.