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Examples of Two State Variables

2 Consider a continuous-time economy with population growth n and constant productivity growth g. The representative household maximizes

$$U = \int_0^T e^{-\rho t} u(C(t))L(t)dt$$

where $\rho > 0$ and u is power utility with elasticity of intertemporal substitution $1/\theta$. Households may invest in domestic capital (K) or in an international capital market. Domestic production $Y = F(K, AL)$ satisfies the usual assumptions. Domestic capital depreciates at rate $\delta > 0$. Let X denote net assets held abroad (so $X < 0$ means borrowing from abroad) and assume foreign assets (or loans) pay an exogenous interest rate $r(t) = \bar{r}$, where $\bar{r} > n + g$. Assume domestic capital can be exchanged for foreign assets instantaneously, so the division of K + X into its components is a choice.

a. Set up the Hamiltonian problem with Z = K + X as the state variable, apply the maximum principle, and derive differential equations for optimal C and Z. Show that K/(AL) is constant.

Answer: Since assets can be optimally divided instantly, we can solve for the ratio of domestic to foreign goods and then use the total as a state variable, Z. We can set up the Hamiltonian by writing the constraint as the following:

$$\dot{Z} = \bar{r}X + F(K, AL) - \delta K - LC \Rightarrow \dot{Z} = \bar{r}Z + F(K, AL) - (\delta + \bar{r})K - LC$$

$$H(C, Z, t, \lambda) = e^{-\rho t} u(C(t))L(t) + \lambda[\bar{r}Z(t) + F(K, AL) - (\delta + \bar{r})K(t) - L(t)C(t)]$$

Taking the first order conditions, we get the following:

$$\frac{\partial H}{\partial C} = e^{-\rho t} u'(C(t))L(t) - \lambda L(t) \Rightarrow \lambda = e^{-\rho t} u'(C(t))$$

$$\frac{\partial H}{\partial K} = \lambda(F_K(K, AL) - (\delta + \bar{r})) = 0 \Rightarrow F_K(K, AL) = (\delta + \bar{r})$$

This basically says that we will adjust domestic capital to equal to foreign return, net of depreciation.

$$\frac{\partial H}{\partial Z} = -\dot{\lambda} = \lambda \bar{r}$$

We have the differential equation in Z from the constraint. Now, we can use the usual tricks to solve for the differential equation in consumption:

$$\frac{\dot{\lambda}}{\lambda} = -\rho + \frac{u''(C)C}{u'(C)} \frac{\dot{C}}{C}$$

$$\Rightarrow \frac{\dot{\lambda}}{\lambda} = -\rho - \theta \frac{\dot{C}}{C}$$

Combining the first-order in consumption with the first-order in the state variable (as we always do), we get

$$-\bar{r} = -\rho - \theta\frac{\dot{C}}{C}$$

$$\Rightarrow \frac{\dot{C}}{C} = \frac{\bar{r} - \rho}{\theta}$$

Thus, we have both differential equations to describe the economy. The last thing that we need to describe this economy is to show that capital is constant over time: recall that our standard assumptions include constant returns to scale. This means that we can write the following:

$$F_K(K, AL) = f'(k) = \delta + \bar{r}$$

We can invert the function f:

$$(f')^{-1}(\delta + \bar{r}) = k^*$$

Thus, we see that capital in efficiency units is constant for all t.

b. Explain why a No-Ponzi condition is needed for the problem to have a solution with finite utility; specify the condition.

Answer:

If we didn't have a No-Ponzi condition, it would be possible for a country to borrow an infinite amount of assets to maximize their utility. This means that it must be the case that the value of foreign assets does not go to negative infinity. We can write this condition in the following way:

$$\lim_{t\to\infty} \lambda(t)X(t) \geq 0$$

c. Suppose $\rho = \bar{r}$ and $X(0) = 0$. Solve for optimal $C(t)$ and $X(t)$ as functions of initial capital $K(0)$.

Answer:

Using our usual terminal conditions, we can integrate the differential equation in total assets:

$$\dot{Z} = \bar{r}Z(t) + F(K, AL) - (\delta + \bar{r})K(t) - L(t)C(t)$$

$$\Rightarrow 0 = Z(0) + \int_0^\infty [A(t)L(t)(f(k^*) - (\delta + \bar{r})k^*) - C(t)L(t)]e^{-\bar{r}t}dt$$

Now, note that if $\rho = \bar{r}$, consumption is constant over time:

$$\frac{\dot{C}}{C} = \frac{\bar{r} - \rho}{\theta} = 0 \Rightarrow C(t) = C(0)$$

Then, we can start subbing in for the growth rate of quantities in our differential equation to solve for consumption in the first time period (note that since X(0) = 0, Z(0) = K(0)):

$$\int_0^\infty C(t)L(t)e^{-\bar{r}t}dt = K(0) + \int_0^\infty [A(t)L(t)(f(k^*) - (\delta+\bar{r})k^*)]e^{-\bar{r}t}dt$$

$$\Rightarrow C(0)L(0)\int_0^\infty e^{-(\bar{r}-n)t}dt = K(0) + [A(0)L(0)(f(k^*) - (\delta+\bar{r})k^*)]\int_0^\infty e^{-(\bar{r}-n-g)t}dt$$

$$\Rightarrow \frac{1}{\bar{r}-n}C(0) = \frac{K(0)}{L(0)} + \frac{1}{\bar{r}-n-g}[A(0)(f(k^*) - (\delta+\bar{r})k^*)]$$

$$C(0) = \frac{K(0)}{L(0)}(\bar{r}-n) + \frac{\bar{r}-n}{\bar{r}-n-g}[A(0)(f(k^*) - (\delta+\bar{r})k^*)]$$

Thus, since $C(t) = C(0)$, we have solved for consumption in every period. We can also solve for foreign assets by using the value of total assets in period T:

$$X(T) = -K(T) + \int_T^\infty [A(t)L(t)(f(k^*) - (\delta+\bar{r})k^*) - C(t)L(t)]e^{-\bar{r}(t-T)}dt$$

Using the same argument as before, we can solve for an explicit quantity:

$$X(T) = -K(T) + \int_T^\infty [A(t)L(t)(f(k^*) - (\delta+\bar{r})k^*) - C(t)L(t)]e^{-\bar{r}(t-T)}dt$$

$$\Rightarrow X(T) = -K(T) - \frac{[A(T)L(T)(f(k^*) - (\delta+\bar{r})k^*)]}{\bar{r}-n-g} + \frac{C(0)L(T)}{\bar{r}-n}$$

$$\Rightarrow \frac{X(T)}{L(T)A(T)} = -k^* - \frac{[f(k^*) - (\delta+\bar{r})k^*]}{\bar{r}-n-g} + \frac{C(0)}{A(T)(\bar{r}-n)}$$

Using what we found for $C(0)$:

$$\frac{X(T)}{L(T)A(T)} = -k^* - \frac{[f(k^*) - (\delta+\bar{r})k^*]}{\bar{r}-n-g} + \frac{\frac{K(0)}{L(0)}(\bar{r}-n) + \frac{\bar{r}-n}{\bar{r}-n-g}[A(0)(f(k^*) - (\delta+\bar{r})k^*)]}{A(T)(\bar{r}-n)}$$

$$\Rightarrow \frac{X(T)}{L(T)A(T)} = [\frac{K(0)}{A(0)L(0)} - k^*] - \frac{A(T)-A(0)}{A(T)}\frac{1}{\bar{r}-n-g}[f(k^*) - (\delta+\bar{r})k^*]$$

1) This question is about an economy with public capital denoted K_G and private capital denoted K_P. Production is give by

$$Y = (K_P)^\alpha (K_G)^\beta (L)^\eta$$

where α, β, $\eta > 0$ and $\alpha+\beta+\eta = 1$. The economy has continuous time, constant population, $L = 1$, constant productivity, and the same depreciation rate, $\delta > 0$ for both K_G and K_P. A representative household owns the production technology and maximizes lifetime utility given by

$$U = \int_0^\infty e^{-\rho t}u(C(t))dt$$

subject to

$$\dot{K}_P = (K_P)^\alpha (K_G)^\beta - \delta K_P - C - T$$

where T are lump sum taxes. Moreover, taxes finance investments in public capital so that:

$$\dot{K}_G = T - \delta K_G$$

.

(a) Assume a government exogenously sets $T = \delta \bar{K}_G$ so that $K_G = \bar{K}_G$ is constant. State the Hamiltonian problem, apply the maximum principle, and derive equations that describe the dynamics of consumption and private capital.

$$H = e^{-\rho t} u(C) + \lambda[(K_P)^\alpha (K_G)^\beta - \delta K_P - C - T]$$

$$\frac{\partial H}{\partial C} = 0 \qquad \Leftrightarrow \qquad e^{-\rho t} u'(C) = \lambda$$

$$\frac{dH}{dK_P} = -\dot{\lambda} \qquad \Leftrightarrow \qquad \frac{\dot{\lambda}}{\lambda} = -[\alpha (K_P)^{\alpha-1} (\bar{K}_G)^\beta - \delta]$$

$$\frac{dH}{d\lambda} = \dot{K}_P \qquad \Leftrightarrow \qquad \boxed{\dot{K}_P = (K_P)^\alpha (K_G)^\beta - \delta K_P - C - T}$$

Now, let's use the first and second conditions to find an equation governing the dynamics of consumption.

$$\Leftrightarrow \ln(u(C)) - \rho t = \ln(\lambda)$$

$$\Rightarrow \frac{u''(C)}{u'(c)} \cdot \dot{C} - \rho = \frac{\dot{\lambda}}{\lambda}$$

$$\Leftrightarrow \frac{u''(C) \cdot C}{u'(C)} \cdot \frac{\dot{C}}{C} = -[\alpha (K_P)^{\alpha-1} (K_G)^\beta - \delta - \rho]$$

$$\Leftrightarrow \boxed{\frac{\dot{C}}{C} = \frac{1}{\theta}[\alpha (K_P)^{\alpha-1} (K_G)^\beta - \delta - \rho]}$$

(b) Explain why (C, K_P) converge to steady state values (C^*, K_P^*) for any initial $K_P(0) > 0$. Construct the phase diagram for this problem. Describe the convergence process from an intial value $K_P(0) > K_P^*$ using both the pgase diagram and a time series plot.

Let's start by looking at the differential equation governing consumption. Notice that because the marginal product of private capital is monotomically decreasing and tends to ∞ as $K_P \to 0$ and tends to 0 as $K_P \to \infty$. This implies that when K_P is small,

$\alpha(K_P)^{\alpha-1}(K_G)^{\beta} - \delta - \rho > 0$ and when K_P is large, $\alpha(K_P)^{\alpha-1}(K_G)^{\beta} - \delta - \rho < 0$. Hence, for small capital, consumption is growing and for large capital consumption is declining. Thus, by the intermediate value theorem, $\exists!\ K_P^*$ s.t. $\dot{C} = 0$. Next, looking at the differential equation for K_P, we see that there is a single, concave $\dot{K}_P = 0$ locus given by $C = (K_P)^{\alpha}(K_G)^{\beta} - \delta K_P - T$. If C is above this locus, then $K_P < 0$ and vice-versa. Moreover, assuming that $K_P = K_P^*$, $\exists!\ C^*$ such that $\dot{K}_P = 0$ because the $\dot{K}_P = 0$ locus is strictly concave. Thus, we see that there is a unique (K_P^*, C^*) such that $\dot{K}_P = \dot{C} = 0$. Furthermore, from our discussion above, it is also evident that the phase diagram for this econonmy is given by:

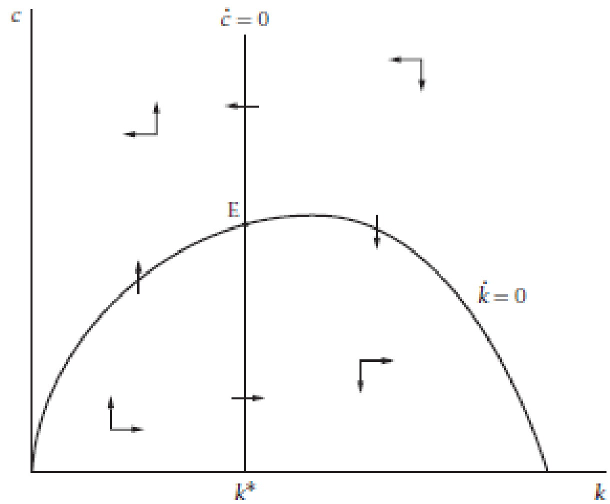
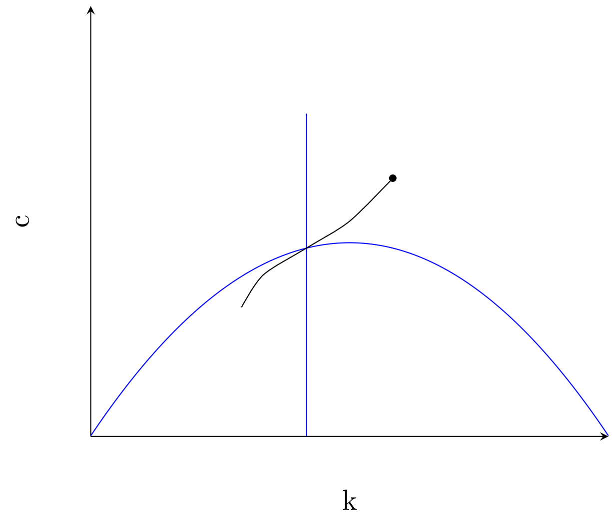


FIGURE 2.3 The dynamics of c and k

Also due to the continuity arguments that we made above, there must also exist a unique saddle path that runs through point E. But why will indivuals place themselves on this saddle path? Because individuals derive no utility from capital directly, we know that they will never place themselves on a path that leads to the horizontal axis (intersection of $\dot{K}_P = 0$ locus and horizontal axis). Moreover, indivuals cannot place themselves on a path that diverges up and to the left due to the fact that this will deplete the capital stock. Once the capital stock is depleted, individuals have nothing to consume. Hence, $\forall\ K_P(0) > 0$, utility maximizing agents will place themselves on the saddle path.

(c) Now assume that the government maximizes household utility by choice of C and T. Also assume that the government can seize private capital and convert it to publi capital and vice-versa. Define total capital as $K = K_P + K_G$.

- Explain briefly why letting the government choose C is an innocuous assumption; and why the problem has K as a single state variable. Because taxes are lump-sum, using the planner's problem in the form of a benevolent dictator will yield identical solutions as solving the household's problem by the Second Welfare Theorem. Moreover, because capital can be seamlessly converted between public and private capital within each period, the only intertemporal savings decision to be made is how much total capital to hold into the next point in time. The optimal capital mix is thus chosen within each period.


c
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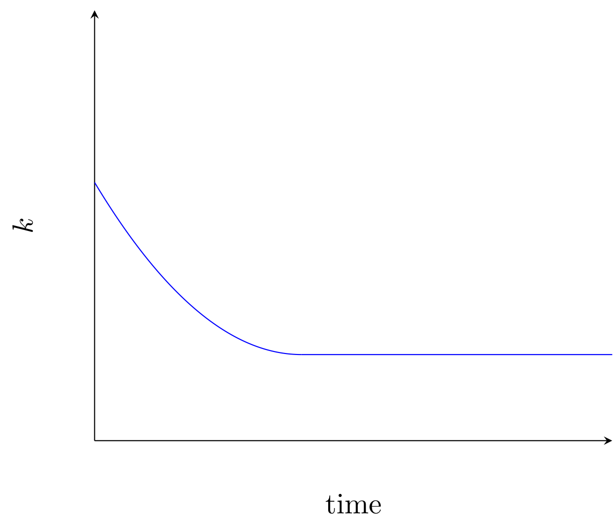
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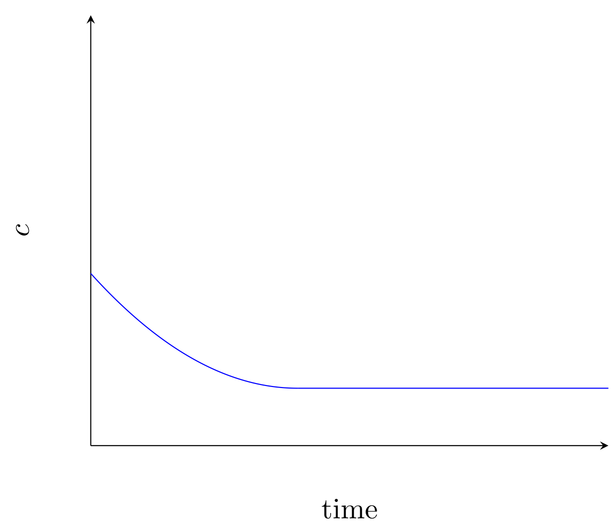
c
time

- State the Hamiltonian problem and apply the maximum principle. Show how total capital is optimally divided into private and public capital. Derive the equations that describe the dynamics of consumption and total capital [You may omit steps that are the same as in part (a)].

  In order to solve this problem, I will first transform the Hamiltonian from part a to have only a single state variable. To do so, I will first find the optimal capital mix. Within each period, the optimal capital mix should be such that the marginal product per dollar price of public and private capital should be equal.

$$\frac{\alpha(K_P)^{\alpha-1}(K_G)^{\beta}}{\alpha(K_P)^{\alpha-1}(K_G)^{\beta}-\delta} = \frac{\beta(K_P)^{\alpha}(K_G)^{\beta-1}}{\beta(K_P)^{\alpha}(K_G)^{\beta-1}-\delta}$$

$$\Leftrightarrow \boxed{K_G = \frac{\beta}{\alpha}K_P}$$

  Next, solve for K_P using the K equation given.

$$K = K_P + K_G$$

$$\Leftrightarrow K = K_P + \frac{\beta}{\alpha}K_P$$

$$\Leftrightarrow \boxed{K_P = \frac{\alpha}{\alpha+\beta}K}$$

$$\Rightarrow \boxed{K_G = \frac{\beta}{\alpha+\beta}K}$$

  Now that we know the optimal private-public capital mix, we can derive the new constraint for the Hamiltonian.

$$K = K_P + K_G$$

$$\Rightarrow \dot{K} = \dot{K}_P + \dot{K}_G$$

$$\Leftrightarrow \dot{K} = (K_P)^{\alpha}(K_G)^{\beta} - \delta K_P - \delta K_G - C$$

$$\Leftrightarrow \boxed{\dot{K} = \left[\frac{\alpha}{\alpha+\beta}\right]^{\alpha} \cdot \left[\frac{\beta}{\alpha+\beta}\right]^{\beta} \cdot K^{\alpha+\beta} - C - \delta K}$$

Now, I'll define $\epsilon = \left[\frac{\alpha}{\alpha+\beta}\right]^{\alpha} \cdot \left[\frac{\beta}{\alpha+\beta}\right]^{\beta}$ so that the Hamiltonian problem becomes

$$H = e^{-\rho t}u(C) + \lambda[\epsilon \cdot K^{\alpha+\beta} - \delta K - C]$$

$$\Rightarrow \frac{dH}{dK} = -\dot{\lambda} \qquad \Leftrightarrow \qquad \frac{\dot{\lambda}}{\lambda} = -\underbrace{\epsilon(\alpha+\beta)}_{\pi} K^{-\eta} + \delta$$

$$\Rightarrow \boxed{\frac{\dot{C}}{C} = \frac{1}{\theta}[\pi K^{-\eta} - \delta - \rho]}$$

- Find conditions for a steady state. Use the notation $(C^{**}, K_P^{**}, K_G^{**}, K^{**})$ with two stars to distinguish from the steady state values in (a).

$$\dot{C} = 0 \qquad \Leftrightarrow \qquad \boxed{K^{**} = \left[\frac{\pi}{\delta+\rho}\right]^{\frac{1}{\eta}}}$$

$$\Rightarrow \qquad \boxed{K_P^{**} = \frac{\alpha}{\alpha+\beta} \cdot \left[\frac{\pi}{\delta+\rho}\right]^{\frac{1}{\eta}}}$$

$$\Rightarrow \qquad \boxed{K_G^{**} = \frac{\beta}{\alpha+\beta} \cdot \left[\frac{\pi}{\delta+\rho}\right]^{\frac{1}{\eta}}}$$

$$\Rightarrow \qquad \boxed{C^{**} = \epsilon K^{**\alpha+\beta} - \delta K^{**}}$$

(d) Assume careless governments have set $K_G = \bar{K}_G < K_G^{**}$ for so long that the economy is in the resulting steady state (K_P^*, C^*). At time t_0, an optimizing government takes over.

- Show how the optimal values $(C^{**}, K_P^{**}, K_G^{**})$ compare to $(C^*, K_P^*, \bar{K}_G)$.
This question is easiest answered by thinking carefully about the steady state capital stock from part (a). In particular, recall that the steady state values from part (a) are given by:

$$C^* = (K_P^*)^{\alpha}(\bar{K}_G)^{\beta} - \delta K_P^* - \delta \bar{K}_G \qquad\qquad K_P^* = \left[\frac{\alpha \bar{K}_G^{\beta}}{\delta+\rho}\right]^{\frac{1}{1-\alpha}}$$

Notice that K_P^* is increasing in $\bar{K}_P$. Thus, since we know that $\bar{K}_G < K_G^{**}$ it must be the case that $K_P^* < K_P^{**}$. Moreover, recall that $K = K_P + K_G$. This implies that if both $K_P^* < K_P^{**}$ and $K_G^* < K_G^{**}$, it must be the case that $K^* < K^{**}$. In turn, if $K^* < K^{**}$ it must also be the case that $C^* < C^{**}$. Intuitively, this should make sense. Households can't be worse off in an "unconstrained" problem as compared to a "constrained" problem. The solution to the "constrained" problem is always an option under the "unconstrained" problem!

- Show how (C, K_P, K_G) change during the convergence process; use a phase diagram and a time series to illustrate your answers. In other words, tell the optimizing government what to do.