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Problem Set 8

Romer Problem 2.17 *Social security in the Diamond model.* Consider a Diamond economy where g is zero, production is Cobb-Douglas, and utility is logarithmic.

(a) *Pay-as-you-go social security.* Suppose the government taxes each young individual an amount T and uses the proceeds to pay benefits to old individuals; thus each old person receives $(1+n)T$.

i. How, if at all, does this change affect equation (2.60) giving k_{t+1} as a function of k_t?

Equation (2.60), with $g = 0$, is given below.

$$k_{t+1} = \frac{(1-\alpha)k_t^\alpha}{(1+n)(2+\rho)}.$$

If we were to go through and add in lump sum taxes, we would determine that

$$k_{t+1} = \frac{(1-\alpha)k_t^\alpha}{(1+n)(2+\rho)} - \frac{T(1+\rho)}{(1+n)(2+\rho)}\left[1 + \frac{1+n}{1+r_{t+1}}\right].$$

ii. How, if at all, does this change affect the balanced-growth-path value of k?

Using the expression from the previous part, we can see that for any lump sum taxes $T > 0$, capital will be lower than when $T = 0$.

iii. If the economy is initially on a balanced growth path that is dynamically efficient, how does a marginal increase in T affect the welfare of current and future generations? What happens if the initial balanced growth path is dynamically inefficient?

Looking at that same expression, if the economy is dynamically efficient, we will be moving the capital stock farther away from the golden rule level when taxes are increased (recall dynamic efficiency implies we are below the golden rule). Doing so will decrease the welfare for the current generation (the young in a given period) and will also decrease the welfare of future generations. On the other hand, if the economy is dynamically inefficient, the taxes can actually move the economy toward the golden rule level, increasing the welfare of future generations.

(b) *Fully funded social security.* Suppose the government taxes each young person an amount T and uses the proceeds to purchase capital. Individuals born at t therefore receive $(1 + r_{t+1})T$ when they are old.

i. How, if at all, does this change affect equation (2.60) giving k_{t+1} as a function of k_t?

Going through similar steps, we can determine that everything that is transferred "nets out."

$$k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}}{(1+n)(2+\rho)}$$

This simple transfer is just going to lead to a one-for-one adjustment in savings. Because the government purchases capital with this "other" savings, there will be no difference in the relationship between k_{t+1} and k_t.

ii. How, if at all, does this change affect the balanced-growth-path value of k?

Because the above relationship between k_{t+1} and k_t is unaffected, this change will not affect the balanced-growth-path level of k.

Problem 4.4 This question is about an overlapping generations economy with a social security system. Individuals have a utility function $U = ln(c_{1t}) + \beta ln(c_{2t+1})$. They supply one unit of labor in the first period of their life. The production technology is Cobb-Douglas with capital share α and 100% depreciation. Let w_t be the wage rate, r_t be the interest rate, and k_t be the capital stock per worker. The population grows at a rate n.

(a) Derive the savings function of young workers and then derive the steady state capital stock in this economy. Under what conditions is r_t greater than or less than n?

As usual, we would begin with deriving the IBC and setting up the Lagrangian. This is identical to what we've seen on the past problem set and in section, so I'm going to simply write down the main results.

$$s_t = \frac{\beta}{1+\beta} \qquad k_{t+1} = \frac{(1-\alpha)\beta}{(1+\beta)(1+n)}k_t^{\alpha} \qquad k^* = \left(\frac{(1-\alpha)\beta}{(1+\beta)(1+n)}\right)^{\frac{1}{1-\alpha}}$$

With the above, we can determine the interest rate in the steady state: $r^* = \alpha k^{*\alpha-1} - 1$.

$$r^* = (1+n)\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta} - 1$$

We can do a little arithmetic to get the following expression

$$\frac{1+r^*}{1+n} = \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta},$$

We can see that $r_t > n$ if the RHS of the last expression exceeds 1.

$$\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta} > 1 \qquad \Longrightarrow \qquad \frac{\alpha}{1-\alpha} > \frac{\beta}{1+\beta}$$

The opposite is true if the inequality above is reversed: $r_t < n$.

(b) Suppose in some period t_0 that the government unexpectedly introduces a pay-as-you-go social security system of the following form: each period, a fraction τ of the wage income is taxed. The tax receipts are immediately distributed to the old generation. Describe the impact of this social security system on the economy (on all variables that you consider relevant). Can you determine the new steady state? Is it important for welfare comparisons whether or not the economy without social security was dynamically efficient? Explain your findings.

Let's begin by finding the new IBC.

First Period: $\quad c_{1t} + a_t = (1-\tau)w_t$

Second Period: $\quad c_{2t+1} = (1+r_{t+1})a_t + (1+n)\tau w_{t+1}$

IBC: $\quad c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = (1-\tau)w_t + \frac{(1+n)\tau w_{t+1}}{1+r_{t+1}}$

The Lagrangian, FOCs, and EE are then

$$\mathcal{L} = ln(c_{1t}) + \beta ln(c_{2t+1}) + \lambda\left[(1-\tau)w_t + \frac{(1+n)\tau w_{t+1}}{1+r_{t+1}} - c_{1t} - \frac{c_{2t+1}}{1+r_{t+1}}\right]$$

$$\frac{\partial\mathcal{L}}{\partial c_{1t}} = 0: \qquad \lambda = \frac{1}{c_{1t}}$$
$$\frac{\partial\mathcal{L}}{\partial c_{2t+1}} = 0: \qquad \lambda = \frac{\beta(1+r_{t+1})}{c_{2t+1}}$$

$$\Longrightarrow \qquad c_{2t+1} = \beta(1+r_{t+1})c_{1t}$$

Plug the EE back into the IBC to obtain an expression for c_{1t}.

$$c_{1t} = \frac{w_t}{1+\beta}\left[(1-\tau) + \frac{(1+n)\tau}{1+r_{t+1}}\right]$$

Note that in deriving the above expression we utilized the fact that, at the steady state, $w_t = w_{t+1}$. Now we can find a_t.

$$a_t = \underbrace{\left[\frac{\beta(1-\tau)}{1+\beta} - \frac{(1+n)\tau}{(1+\beta)(1+r_{t+1})}\right]}_{s_t} w_t$$

We see in the above expression, that moving from $\tau = 0$ to something strictly positive, there will be an unambiguous drop in assets (and the savings rate). To get into more detail about capital, take the expression for K_{t+1} and put it into effective units.

$$K_{t+1} = L_t a_t \quad \Longrightarrow \quad k_{t+1} = \frac{a_t}{1+n} \quad \Longrightarrow \quad k_{t+1} = \left[\frac{\beta(1-\tau)}{(1+\beta)(1+n)} - \frac{\tau}{(1+\beta)(1+r_{t+1})}\right] w_t$$

Plugging in for our Cobb-Douglas functional form, we can find the steady state for k by setting $k_t = k_{t+1}$.

$$k = \left[\frac{\beta(1-\tau)}{(1+\beta)(1+n)} - \frac{\tau}{(1+\beta)(\alpha k^{\alpha-1})}\right](1-\alpha)k^{\alpha}$$

$$k^* = \left\{\frac{(1-\alpha)\beta(1-\tau)}{(1+\beta)(1+n)}\left[1 + \frac{(1-\alpha)\tau}{\alpha(1+\beta)}\right]^{-1}\right\}^{\frac{1}{1-\alpha}}$$

Thus for any increase in τ, the steady state capital stock will be unambiguously lower. If the economy is dynamically inefficient, this could be welfare improving, as the economy is being pushed toward the golden rule level of capital. Under dynamic *efficiency*, however, the social security program will lead to welfare losses.

Problem 4.5 Consider the following overlapping generations economy. Individuals in generation t maximize utility $U = ln(c_{1t}) + \beta ln(c_{2t+1})$. They work one unit when young, earning a wage w_t, and save an amount a_t. Technology is Cobb-Douglas, with capital share α and 100% depreciation, so output net of depreciation is $Y_t = K_t^{\alpha} L_t^{1-\alpha} - K_t$. The number of individuals in generations t is L_t. The size L_t of generations t grows at a fixed rate $n > 0$, $L_{t+1} = (1+n)L_t$. There is also a government that operates a pay-as-you-go social security system: each period, the young pay a tax

$T_t = \tau w_t$, $0 < \tau < 1$. (Since labor supply is fixed, the tax is lump sum.) The receipts are given to the old as a transfer TR_t.

(a) Set up the individual optimization problem for generation t, derive the first order conditions, and derive the savings function.

First Period: $\quad c_{1t} + a_t = w_t - T_t$

Second Period: $\quad c_{2t+1} = (1 + r_{t+1})a_t + TR_{t+1}$

IBC: $\quad c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}} = w_t - T_t + \frac{TR_{t+1}}{1 + r_{t+1}}$

Noting that $T_t = \tau w_t$ and $TR_{t+1} = (1 + n)\tau w_{t+1}$. The Lagrangian and associated FOCs look very similar to what we've seen previously.

$$\mathcal{L} = ln(c_{1t}) + \beta ln(c_{2t+1}) + \lambda \left[w_t - T_t + \frac{TR_{t+1}}{1 + r_{t+1}} - c_{1t} - \frac{c_{2t+1}}{1 + r_{t+1}} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0 : \qquad \lambda = \frac{1}{c_{1t}}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0 : \qquad \lambda = \frac{\beta(1 + r_{t+1})}{c_{2t+1}}$$

$$\implies \qquad c_{2t+1} = \beta(1 + r_{t+1})c_{1t}$$

Plug the EE back into the IBC to obtain an expression for c_{1t} and, then, a_t. To make this easier, plug in for both T_t and TR_{t+1} (notice the savings function is the same as in the last problem).

$$a_t = \underbrace{\left[\frac{\beta(1 - \tau)}{1 + \beta} - \frac{(1 + n)\tau}{(1 + \beta)(1 + r_{t+1})} \right]}_{s_t} w_t$$

(b) Explain why the ratio of transfers to wages is $TR_t/w_t = \tau(1 + n) \equiv b$.

Transfers are equal total wages in the economy multiplied by the tax rate, τ, and again multiplied by the increase in the population because more people are being taxed than are being transferred to. Thus, $TR_t = \tau(1 + n)w_t$. Dividing by w_t then gives us $b = \tau(1 + n)$.

(c) Derive an equation linking next period's capital labor ratio $k_{t+1} = K_{t+1}/L_{t+1}$ to the current capital labor ratio k_t, and determine the steady state.

$$K_{t+1} = L_t a_t \qquad \Longrightarrow \qquad k_{t+1} = \frac{a_t}{1+n} \qquad \Longrightarrow \qquad k_{t+1} = \left[\frac{\beta(1-\tau)}{(1+\beta)(1+n)} - \frac{\tau}{(1+\beta)(1+r_{t+1})}\right] w_t$$

Notice that this is the same as in 4.4. Instead of reinventing the wheel, the final result when solving for the steady state is reproduced below.

$$k^* = \left\{\frac{(1-\alpha)\beta(1-\tau)}{(1+\beta)(1+n)}\left[1+\frac{(1-\alpha)\tau}{\alpha(1+\beta)}\right]^{-1}\right\}^{\frac{1}{1-\alpha}}$$

For parts (d)-(f) suppose that the economy is in the steady state at some date t_0. Then population growth unexpectedly stops: in period t_0 everyone learns that L_t will be constant for all $t \geq t_0$.

(d) Suppose the tax rate τ is held constant; transfers are varied, if necessary. Determine how the change in population growth affects the time path of the capital stock.

Setting $n = 0$ in the equation for the relationship for capital.

$$k_{t+1} = \left[\frac{\beta(1-\tau)}{1+\beta} - \frac{\tau}{(1+\beta)(1+r_{t+1})}\right] w_t$$

We can see immediately that k will be higher. When people realize that their income will be lower in the future (from transfers), they will save more. We could do the same thing for k^*. Thus after t_0, the capital stock will increase to it's new steady state level $k^{*'}$:

$$k^{*'} = \left\{\frac{(1-\alpha)\beta(1-\tau)}{1+\beta}\left[1+\frac{(1-\alpha)\tau}{\alpha(1+\beta)}\right]^{-1}\right\}^{\frac{1}{1-\alpha}}.$$

(e) Consider the same change in population growth, but assume that the ratio of transfers to wages, b is held constant. Again, determine how the change in population growth affects the time path of k.

First, rewrite our expression for k_{t+1} in terms of b and set $n = 0$ elsewhere.

$$k_{t+1} = \left[\frac{\beta(1-\tau)}{1+\beta} - \frac{b}{(1+\beta)(1+r_{t+1})}\right] w_t$$

And so we'll have

$$k^{*''} = \left\{ \frac{(1-\alpha)\beta(1-\tau)}{1+\beta} \left[1 + \frac{(1-\alpha)b}{\alpha(1+\beta)} \right]^{-1} \right\}^{\frac{1}{1-\alpha}}.$$

The new steady state will be lower relative to part (d) because payments by the young (to the old) are relatively higher, and so less can be saved and put into capital.

(f) Compare how the alternative policy responses in (d) and (e): how do the capital stocks differ? Can you tell which different generations are better of with one or the other policy?

Comparing $k^{*'}$ and $k^{*''}$ above, we can see that the capital stock from part (e) will be lower than in (d). Without knowledge of whether or not the economy is dynamically efficient, however, we cannot comment on the welfare in either event. Notice also that, for the old people at the time of the change, the policy in (e) is strictly preferred. The old in (d) are unambiguously made worse off because they are transferred less than what they had anticipated. On the other hand, the young in (e) can be worse than those in (d).

Problem 4.6 Consider the following overlapping generations economy. Individuals in generation t maximize a power utility function

$$U = \frac{c_{1t}^{1-\theta}}{1-\theta} + \beta \frac{c_{2t-1}^{1-\theta}}{1-\theta},$$

with $\beta \in (0,1)$ and $\theta > 0$. Cohort size grows over time at some rate n_t, that may vary over time: $L_t = L_{t-1}(1+n_t)$ (assume L_0 is given). Output is given by a linear production function $Y_t = rK_t + wL_t$, where r and w are positive constants. In period 0, capital $K_0 > 0$ is held by the old. Let (T_{1t}, T_{2t}) denote period-t net taxes on the young and old. Let a_t denote individual assets. Depreciation is zero; productivity is constant; there is no government spending and no government debt.

(a) Set up the optimization problem of a young individual in period t for arbitrary (T_{1t}, T_{2t}). Derive the optimality conditions. Explain how a_t depends on wages, taxes, and interest rates.

First Period: $c_{1t} + a_t = w - T_{1t}$

Second Period: $c_{2t+1} = (1+r)a_t - T_{2t+1}$

IBC: $c_{1t} + \frac{c_{2t+1}}{1+r} = w - T_{1t} - \frac{T_{2t+1}}{1+r}$

The Lagrangian, FOCs, and EE are then

$$\mathcal{L} = \frac{c_{1t}^{1-\theta}}{1-\theta} + \beta \frac{c_{2t-1}^{1-\theta}}{1-\theta} + \lambda \left[w - T_{1t} - \frac{T_{2t+1}}{1+r} - c_{1t} - \frac{c_{2t+1}}{1+r} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0 : \qquad \lambda = c_{1t}^{-\theta}$$
$$\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0 : \qquad \lambda = \beta(1+r) c_{2t+1}^{-\theta}$$

$$\implies \qquad c_{2t+1} = [\beta(1+r)]^{\frac{1}{\theta}} c_{1t}$$

Plug the EE into the IBC to obtain an expression for c_{1t}.

$$c_{1t} = \frac{1}{1 + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1-\theta}{\theta}}} \left[w - T_{1t} - \frac{T_{2t+1}}{1+r} \right].$$

Assets are thus

$$a_t = \frac{1}{1 + \beta^{\frac{1}{\theta}} (1+r)^{\frac{1-\theta}{\theta}}} \left[\beta^{\frac{1}{\theta}} (1+r)^{\frac{1-\theta}{\theta}} (w - T_{1t}) + \frac{T_{2t+1}}{1+r} \right].$$

Assets are increasing in wages, decreasing in taxes when young (less is available to save), and increasing in taxes when old (higher taxes later in life lowers income in the future and encourages more saving today. On the other hand, assets are ambiguously affected by the interest rate. This is because we do not know the relative sizes of the income and substitution effects. That is, the answer will depend on the level of r and the value of θ.

(b) Consider a simple social security system: taxes on workers are a constant $\tau > 0$, $T_{1t} = \tau < w$. Transfers to retirees are $T_{2t} = \tau(1+n)$, where $n_t = n$ is constant and $n < r$. Explain how social security affects asset accumulation. Does the economy have a steady-state? How fast is convergence? Explain your findings?

Plug the above information into the IBC.

$$c_{1t} + \frac{c_{2t+1}}{1+r} = w - \tau + \frac{(1+n)\tau}{1+r}$$

The FOCs are the same. Skipping ahead to the expression for a_t (which we need to determine the evolution of k over time) ...

$$a_t = \frac{1}{1+\beta^{\frac{1}{\theta}}(1+r)^{\frac{1-\theta}{\theta}}}\left[\beta^{\frac{1}{\theta}}(1+r)^{\frac{1-\theta}{\theta}}(w-\tau) - \frac{(1+n)\tau}{1+r}\right]$$

Continuing, recall that $K_{t+1} = L_t a_t$.

$$k_{t+1} = \frac{a_t}{1+n} \qquad \Longrightarrow \qquad k_{t+1} = \frac{1}{1+\beta^{\frac{1}{\theta}}(1+r)^{\frac{1-\theta}{\theta}}}\left[\frac{\beta^{\frac{1}{\theta}}(1+r)^{\frac{1-\theta}{\theta}}(w-\tau)}{1+n} - \frac{\tau}{1+r}\right]$$

Importantly, w and r are exogenous and constant, meaning $k_{t+1} = k_t = k^*$ for all t because it is pinned down directly by model parameters (and not endogenous variables). That is, convergence is instant, and the level of capital will be some constant level given by the expression above.

(c) Suppose that population growth declines: $n_t = n^*$ for $t \leq t_0$ and $n_t = n^{**}$ for $t \geq t_0 + 1$. Assume that taxes on workers are unchanged at $\tau > 0$, and let $T_{2t} = \tau(1+n_t)$ to vary. How does the decline in population growth affect a_t? (When does the impact start?) Which generations experience higher or lower utility than without reduced population growth.

The drop in population growth will serve to reduce the lifetime wealth of all generations, limiting their ability to consume. Because τ is constant, they are paying the same amount in taxes out of their fixed income w, but are receiving less when old because there are less people with which to be transferred money from. We can exploit our earlier answer for a_t to determine how a_t will be affected. A lower n can be seen to increase the assets saved by the young. The effect on savings happens immediately.

Problem 4.7 This question is about an overlapping generations economy in which there are durable goods that produce earnings–to be specific, call them fruit trees. Except for the fruit trees, the assumptions are standard: individuals have log-utility with time preference parameter β. They consume and work in their first period of life and receive a wage w_t. They consume but do not work in the second period. Firms produce output from capital and labor at constant returns to scale. The capital share is $\alpha \in (0, 1)$ and the depreciation rate is 100%. To simplify, assume that there is no population growth and no government.

The number of fruit trees and their yield is exogenous. Trees are infinitely lived. The number of fruit trees equals the number of individuals per generation. Initially, each member of the old generation owns one tree. At the start of each period, a tree yields ε units of output (fruits) to its owner, where $\varepsilon > 0$ is a constant. After the harvest, a market opens where the old sell trees to the young. Let x_t be the number of fruit trees that each member of the young generation buys at time t, where $x_t \geq 0$. Let p_t be the period-t market price (assume that it's positive). In period $t+1$, when the period-t young are old, each person receives ε per tree and then sells the tree at a price $p_{t+1} > 0$ to the next generation.

Let $a_t^k \geq 0$ be the amount of savings that each member of the young generation supplies to the capital market. Let $a_t = x_t p_t + a_t^k$ be total individual savings. Individuals take wages, interest rates, and tree prices, as given.

(a) For reference, derive the steady state capital stock k^* and the steady state interest rate r^* for the economy *without* fruit trees. I.e. the standard OG model. Show that the economy is dynamically inefficient for some value of α.

Because we have done this before, only the main results are produced below (it's expected that you can derive all of this; you should not simply skip to memorized expressions).

$$k^* = \left(\frac{(1-\alpha)\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}} \qquad\qquad 1 + r^* = \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}$$

Recall from class that dynamic efficiency is implied if $r^* > n$ (i.e. $1 + r > 1 + n$). Thus we'll need

$$n < \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta} - 1$$

for the economy to be dynamically efficient. Because $n = 0$, this can be simplified to

$$\frac{1+\beta}{\beta} > \frac{1-\alpha}{\alpha}.$$

Briefly, this lower bound on r can be determined by figuring out what is necessary for the economy to be Pareto efficient (Professor Bohn's slides are pretty good on this).

(b) Specify the individual's optimization problem with fruit trees. Show that individuals will not by trees *and* supply saving to the capital market, unless the rate of return on capital r_{t+1} and equilibrium prices of fruit trees satisfy the condition

$$p_t = \frac{\varepsilon + p_{t+1}}{1 + r_{t+1}}. \qquad (\star)$$

First Period: $c_{1t} + a_t^k + p_t x_t = w_t$

Second Period: $c_{2t+1} = (1 + r_{t+1})a_t^k + \varepsilon x_t + p_{t+1} x_t$

IBC: $c_{1t} + \dfrac{c_{2t+1}}{1+r} = w_t + \dfrac{(\varepsilon + p_{t+1})x_t}{1 + r_{t+1}} - p_t x_t$

Now we want to show when agents will save in both markets under ($\star$). Ex ante, this should be obvious because agents obtain utility of consumption, and both investment vehicles enable this consumption. Because the end result is future consumption, what matters is the return per unit cost of these vehicles. If standard savings deliver a greater "bang for your buck", then the agent will save. If fruit trees are more fruitful (pun intended) than savings, the agent will invest in these trees.

Thus, we will want to compare the rate of return per unit cost of each.

$$ROR_{trees} = \frac{\epsilon + p_{t+1}}{p_t} \qquad\qquad ROR_{saving} = (1 + r_{t+1})$$

$$\Rightarrow \frac{\epsilon + p_{t+1}}{p_t} = (1 + r_{t+1})$$

$$\Leftrightarrow p_t = \frac{\epsilon + p_{t+1}}{1 + r_{t+1}}$$

If the above expression does not hold, then individuals will either buy only trees or only save.

(c) Assuming ($\star$) holds, derive the individual demand functions for a_t, c_{1t}, and c_{2t+1}. Specify the economy's equilibrium conditions.

With the typical log-utility specification, the Lagrangian, FOCs, and EE will be the same as we have seen in other problems. Moving forward, plug in $c_{2t+1} = \beta(1 + r_{t+1})c_{1t}$ into the IBC that we found in part (b) to begin solving for these functions. First solve for consumption in both periods.

$$c_{1t} = \frac{1}{1+\beta}\left[w_t + \frac{(\varepsilon + p_{t+1})x_t}{1 + r_{t+1}} - p_t x_t\right] = \frac{1}{1+\beta} w_t$$

$$c_{2t+1} = \frac{\beta(1 + r_{t+1})}{1+\beta}\left[w_t + \frac{(\varepsilon + p_{t+1})x_t}{1 + r_{t+1}} - p_t x_t\right] = \frac{\beta(1 + r_{t+1})}{1+\beta} w_t$$

Now, solve for assets by plugging c_{1t} into the first period budget constraint.

$$a_t = \frac{1}{1+\beta}\left[\beta w_t + \frac{(\varepsilon + p_{t+1})x_t}{1 + r_{t+1}} - p_t x_t\right] = \frac{\beta}{1+\beta} w_t$$

The equilibrium condition of the OG model requires that markets clear. Namely that the capital demanded by firms equals the amount saved (in capital savings) by agents: $K_{t+1} = L_t a_t^k$.

(d) Derive conditions for the steady state interest rate r^*, the steady state capital stock k^*, and the steady state price of fruit trees p^*. Can you show that this economy is always dynamically efficient? [Hint: can you show that the equilibrium conditions imply $r^* > 0$?]

$$a_t = \frac{\beta}{1+\beta} w_t \qquad \Rightarrow a_t^k = \frac{\beta}{1+\beta} - p_t x_t$$

$$\Rightarrow k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha - p_t x_t$$

Note that because population is not growing, the stock of fruit trees is exogenous and constant

$$\Rightarrow \boxed{k^* = \frac{\beta}{1+\beta}(1-\alpha)k^{*\alpha} - p^* x}$$

By the no arbitrage condition we get

$$\boxed{p^* = \frac{\epsilon + p^*}{1+r^*}}$$

From the production function we ge

$$\boxed{r^* = \alpha k^{*\alpha - 1}}$$

Thus, we have pinned down equilibrium capital and prices. Moreover, we can rearrange our second equilibrium condition:

$$p^* = \frac{\epsilon + p^*}{1 + r*} \qquad \Rightarrow r^* = \frac{\epsilon + p^*}{p^*} - 1$$

$$\Leftrightarrow \boxed{r^* = \frac{\epsilon}{p^*}}$$

Thus, because we assume that both dividends and the equilibrium price of trees are strictly positive, we know that $0 < r^* < \infty$. Since the aggregate economy is not growing this condition implies that the economy is always dynamically efficient.