Econ 204B: Section 3

Ryan Sherrard

University of California, Santa Barbara

January 24, 2019

Introduction to Recursive Macroeconomics

- Very useful / intuitive paradigm in modeling dynamic macroeconomic problems
- As Ljungqvist & Sargent calls it, we will delve into the imperialism of the recursive method known as *dynamic programming*
- The key "notion" underlying this method is a finite dimensional *state* that, at any point in time, fully characterizes all information necessary to make *choices*
- The above might seem innocuous, but is very important: keeping track of an infinite history of all variables is problematic
- To begin we'll assume that the state evolves deterministically; we'll very quickly see what happens when there is uncertainty about the future

- To get a feel about what a recursive problem looks like, let's jump right into it; let's write down the model we saw last section *recursively*

- After getting a feel for what one of these looks like, we'll cover some fairly technical mathematical results that underlie why this formulation works

- In doing so, it should become clear how flexible the generic model is, and just what we can do with it

- Moving forward, we will learn different ways to solve these problems and encounter various applications of the theory

Writing a Simple Model Recursively

Let's revisit the model from last week's section and rewrite it recursively. If you'll recall, it was a simple, deterministic planner's problem where a representative agent maximized the present discounted value of lifetime utility over consumption.

What we want is to write an expression for the value of being in some state using something like an indirect utility function (I say "like" because the objective is not always a utility function). That is, it will be endowed with a idea of "optimized value."

Because we don't want an infinite sum to deal with, we are going to focus on writing the expression in terms of the objective today and a continuation value associated with everything happening from tomorrow on.

$$V(k) = \max_{c,a,k'} \left\{ u(c) + \beta V(k') \right\} \qquad \text{s.t.} \qquad \begin{aligned} c + a &\leq f(k) \\ k' &= (1-\delta)k + a \\ c, k &\geq 0 \end{aligned}$$

For clarity, a common notation has variable values *today* be simply the letter, while a "prime" is added for the value of a particular variable *tomorrow*. Notice also that I didn't write that k_0 was given: it's essentially implied by the way the problem is written.

The value of being in state k today is the value I will get directly from my current-period objective, plus the value I will obtain from being in the state k' tomorrow (discounted of course).

For now, we'll just take the fact that V on the LHS is the *same* V as that on the right hand side. We'll need more results to establish when that is (and isn't) the case.

This problem is basic enough such that we don't need any "fancy" solutions methods to tackle it (we'll see those in due time). First, let's substitute out some of our constraints.

$$a = k' - (1-\delta)k \qquad \Longrightarrow \qquad c \leq f(k) + (1-\delta)k - k'$$

(Oft-unstated:) Let's assume that consumers are locally non-satiated (which will be captured by any standard utility function); the budget constraint above will bind. We thus have

$$V(k) = \max_{k'} \left\{ u\left(f(k) + (1-\delta)k - k'\right) + \beta V(k') \right\}.$$

Above is just a simple optimization problem. The FOC is

$$\frac{dV(k)}{dk'} = 0: \qquad -\frac{du(\cdot)}{dc} + \beta \frac{dV(k')}{dk'} = 0.$$

And so we meet our first roadblock. What is $dV(k')/dk'$? We will have to utilize an envelope theorem (namely the Benveniste & Sheinkman theorem). We've actually applied this theorem before (in last week's section), and we'll revisit the theorem in greater detail later.

For now, note that because V is the same on both sides (partly a product of it being an infinite horizon), we can find $dV(k)/dk$ and "push forward" the result one period.

$$\frac{dV(k)}{dk} = \frac{du(\cdot)}{dc}[\underbrace{f'(k) + 1 - \delta}_{1+r}] \qquad \Longrightarrow \qquad \frac{dV(k')}{dk'} = \frac{du(\cdot')}{dc'}[1 + r']$$

Plug the above result into the FOC and you'll see a familiar friend.

$$\frac{du(\cdot)}{dc} = \beta(1 + r')\frac{du(\cdot')}{dc'} \tag{EE}$$

- From here, we can make functional form specifications, etc.
- Also note that we can do an iterative process like last time on the EE to get the policy function; if log utility and Cobb-Douglas production, then

$$k' = \alpha\beta k^{\alpha}$$

- While the above might not seem much easier than what we saw last time, remember . . .
  - we didn't have to deal with an infinite number of constraints
  - this is still a pretty basic model
- Now with a basic understanding of what we're working with, let's put this aside and "open up the hood" to see what's going on

Mathematical Preliminaries

The tool that we are using, the *Bellman Equation*, is a *functional* equation (i.e. it maps the space of functions into functions) and generally takes on the form

$$V(x) = sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta V(y) \right\},$$

where x is a finite dimensional vector we interpret as the *state*, y the *choice*, and $F(x, y)$ is some "current period" objective. (Usually we will have a unique solution, in which case you might see "max" instead of "sup.") Γ denotes the set of choices available given the state x.

Our ultimate goal is to solve for V. If you look at the above you might be reminded of a fixed point result. Indeed, the V we will want is exactly that, a fixed point.

Big Picture / "Road Map"

1. Lay out the fixed point theorem we want to use (we want to prove V exists): **The Contraction Mapping Theorem**

2. Prove that the Bellman Operator satisfies that fixed point theorem: **Blackwell's Sufficiency Conditions**

3. Generalize our results to handle state-choice relationships (i.e. how the current period state maps into the feasible choices and their associated returns) as broadly as possible: **The Theorem of the Maximum**

4. Mention an often-utilized envelope theorem: **Benveniste-Scheinkman Theorem**

1. The Contraction Mapping Theorem

Defnition. Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself. T is a *contraction mapping* (with modulus β) if for some $\beta \in (0, 1)$, we have that $\rho(Tx, Ty) \leq \beta\rho(x, y)$ for all $x, y \in S$.

We call the function T a *contraction mapping* if, when applied to x and y, the distance gets "closer."

Theorem 3.2. *Contraction Mapping Theorem.* If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then

- T has exactly one fixed point V in S, and
- for any $V_0 \in S$, $\rho(T^n V_0, V) \leq \beta^n \rho(V_0, V)$, $n = 0, 1, 2, \ldots$.

There will be a fixed point of T, which we will call V and that, starting from anywhere in S, if we keep applying T to our result we will converge to the true V eventually.

2. Blackwell's Sufficiency Conditions

Theorem 3.3 *Blackwell's sufficient conditions for a contraction.* Let $X \subseteq \mathbb{R}^l$, and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying

- (monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $Tf(x) \leq Tg(x)$, for all $x \in X$;
- (discounting) there exists some $\beta \in (0, 1)$ such that $[T(f + a)](x) \leq Tf(x) + \beta a$, all $f \in B(X)$, $a \geq 0$, $x \in X$.

Note: $(f + a)(x) = f(x) + a$.

Problems arise when trying to prove that an operator is a contraction. The above two sufficiency conditions, however, allow us to quickly (and easily) verify if the operator we are using is indeed a contraction (and so has a fixed point). While it won't rule out if our operator isn't a contraction, it can be used to tell us if it is. Now, let's see if our Bellman equation is a contraction ...

Proof.

Let the Bellman Operator take the form $Tf(x) = h(x, y) + \beta f(y)$ where we are maximizing w.r.t y. W.L.O.G. let $f(x) \leq g(x)$, then

$$Tf(x) = h(x, y) + \beta f(y)$$
$$Tg(x) = h(x, y) + \beta g(y)$$

$$\begin{aligned} Tf(x) - Tg(x) &= [h(x, y) + \beta f(y)] - [h(x, y) + \beta g(y)] \\ &= \beta[f(y) - g(y)] \end{aligned}$$

Since we know that $f(x) \leq g(x)$, we know that $Tf(x) \leq Tg(x)$. Thus, the Bellman Operator is monotonic. Next, we can write

$$\begin{aligned} T(f + a)(x) &= h(x, y) + \beta(f + a)(y) \\ &= h(x, y) + \beta f(y) + \beta a \\ &= [h(x, y) + \beta f(y)] + \beta a \\ &= Tf(x) + \beta a. \end{aligned}$$

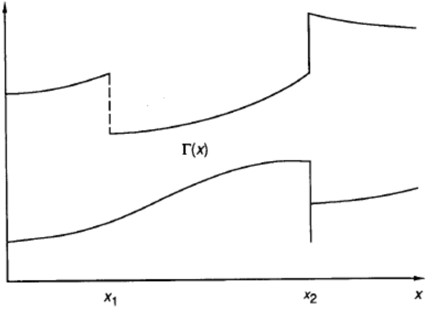
Thus, the Bellman Operator discounts. By Blackwell's sufficiency conditions, we have that the Bellman Operator is a contraction mapping. □

The Theorem of the Maximum

Definition. A correspondence $\Gamma : X \to Y$ is *lower hemi-continuous* (l.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, there exists $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \to y$ and $y_n \in \Gamma(x_n)$, all $n \geq N$. [If $\Gamma(x')$ is nonempty for all $x' \in X$, then it is always possible to take $N = 1$.]

Definition. A compact-valued correspondence $\Gamma : X \to Y$ is *upper hemi-continuous* (u.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$ and every sequence $\{y_n\}$ such that $y_n \in \Gamma(x_n)$, all n, there exists a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(X)$.

Definition. A correspondence $\Gamma : X \to Y$ is *continuous* at $x \in X$ if it is both u.h.c. and l.h.c. at x.



Visualizing these continuity concepts in 2-dimensions (from SLP).

- at x_1: l.h.c. but not u.h.c.
- at x_2: u.h.c. but not l.h.c
- all other x: l.h.c. and u.h.c. (i.e. continuous)

Theorem 3.6. *Theorem of the Maximum.* Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$, let $f : X \times Y \to \mathbb{R}$ be a continuous function, and let $\Gamma : X \to Y$ be a compact-valued and continuous correspondence. Then the function $h : X \to \mathbb{R}$ (where $h(x) = \max_{y \in \Gamma(x)} f(x, y)$) is continuous and the correspondence $G : X \to Y$ (where $G(x) = \{y \in \Gamma(x) | f(x, y) = h(x)\}$) is nonempty, compact-valued, and u.h.c.

Colloquially, we'll need the following assumptions about the correspondence: it must be non-empty, compact-valued, and continuous. We trivially assume the first. Further, we can guarantee that we have a compact-valued and continuous correspondence if we assume that the objective function (usually a utility function) is bounded, continuous, and strictly concave. If all of this holds (that is, these together with the operator being a contraction) then there will exist a fixed point of our Bellman Equation.

Benveniste-Scheinkman Theorem

Theorem 4.10. *Benveniste-Scheinkman.* Let $X \subseteq \mathbb{R}^k$ be a convex set, let $V : X \to \mathbb{R}$ be concave, let $x_0 \in int\, X$, and let D be a neighborhood of x_0. If there is a concave, differentiable function $W : D \to \mathbb{R}$, with $W(x_0) = V(x_0)$ and with $W(x) \leq V(x)$ for all $x \in D$, then V is differentiable at x_0, and

$$V_i(x_0) = W_i(x_0), \quad i = 1, 2, \ldots, l.$$

What does this mean for us? Recall that our simple problem.

$$\underbrace{V(k)}_{V(x)} = \max\{u(c) + \beta \underbrace{V(k')}_{W(x)}\}$$

Well, now that we know that there is a fixed point that solves the above functional equation, we can be sure that V evaluated at k_0 on the LHS is equal to the RHS V evaluated at k_0.

We also know that $V(k') \leq V(k)$. Thus, we'll have that

$$\frac{dV(k)}{dk} = \frac{dV(k')}{dk'}$$

In terms of these problems, this is to say that we can use this envelope theorem to take a derivative and "push forward" a period. You might also see this taken one step further. Note that in our Bellman Equation that $V(k')$ is not a function of k (any information contained in k is captured in k'). Thus we know that

$$\frac{dV(k)}{dk} = \frac{du(c)}{dc}\frac{dc}{dk} + \beta \cancelto{0}{\frac{dV(k')}{dk}}$$

$$= \frac{du(c)}{dc}\frac{dc}{dk}.$$

Coming Attractions . . .

- Moving forward, we will formalize the 3 main solution techniques that we will be using; (in no particular order) they are
  1. Functional Euler Equation
  2. Guess and Verify (a.k.a the Method of Undetermined Coefficients)
  3. Value / Policy Function Iteration
- Iterative methods, as might be obvious, are pretty naturally extended to the use of a computer (but a lot can still be done with them analytically)
- We will see more examples of problems that can be cast in a recursive framework; examples include adding uncertainty into the mix, having heterogeneous agents, etc.