ECON 204B Midterm

1) (30 points) Recall Blackwell's Sufficiency Conditions for a contraction:

   i Monotinicity: T is monotone if for $f(x) < g(x)\ \forall x \in X$, then:

$$Tf(x) \leq Tg(x)\ \forall x \in X$$

   ii Discounting: T discounts if for some $\beta \in (0, 1)$ and any $a \in \mathbb{R}$:

$$T(f + a)(x) \leq Tf(x) + \beta a\ \forall x \in X$$

   Prove that the Bellman Operator satisfies both of these conditions.

   Let the Bellman Operator take the form $Tf(x) = h(x, y) + \beta f(y)$ where we are maximizing w.r.t y. W.L.O.G. let $f(x) \leq g(x)$, then

$$\begin{aligned} Tf(x) &= h(x, y) + \beta f(y) \\ Tg(x) &= h(x, y) + \beta g(y) \end{aligned}$$

$$\begin{aligned} Tf(x) - Tg(x) &= [h(x, y) + \beta f(y)] - [h(x, y) + \beta g(y)] \\ &= \beta[f(y) - g(y)] \end{aligned}$$

   Since we know that $f(x) \leq g(x)$, we know that $Tf(x) \leq Tg(x)$. Thus, the Bellman Operator is monotonic. Next, we can write

$$\begin{aligned} T(f + a)(x) &= h(x, y) + \beta(f + a)(y) \\ &= h(x, y) + \beta f(y) + \beta a \\ &= [h(x, y) + \beta f(y)] + \beta a \\ &= Tf(x) + \beta a. \end{aligned}$$

   Thus, the Bellman Operator discounts. By Blackwell's sufficiency conditions, we have that the Bellman Operator is a contraction mapping.

2) (35 Points) Suppose you are unemployed and trying to decide if you should enter the labor force. You have preferences over consumption and search effort and solve the following problem:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t) - a_t\right] \qquad t \geq 0 \qquad 0 < \beta < 1$$

   with $c_t, a_t \geq 0$. An unemployed individual searching with effort a receives a job with probability $p(a) \in [0, 1]$ with $p(0) = 0$. Assume $p(\cdot)$ is increasing, strictly concave, and twice differentiable. There is no borrowing or saving.

a) (10 Points) Suppose that once you become employed you receive wage w forever (i.e. no quits or fires). Write down the value, V^e of being employed.

$$V^e = \max_a \sum_{t=0}^{\infty} \beta^t [U(w) - a]$$

$$\Rightarrow a^e = 0$$

$$\Rightarrow V^e = \frac{U(w)}{1-\beta}$$

b) (15 Points) Denote the value of an unemployed individual as V^u. Write down the Bellman equation for an unemployed worker. What are the state and choice variables?

$$V^u = \max_{a \geq 0} \{U(0) - a + \beta[p(a)V^e + (1 - p(a)V^u]$$

Note that here, the only state variable is your employment state and the only choice variable is your search effort.

c) (10 Points) Find an expression for the optimal search effort.

$$V^u = \max_{a \geq 0} \{U(0) - a + \beta[p(a)V^e + (1 - p(a)V^u]$$

$$\Rightarrow \beta p'(a)[V^e - V^u] \leq 1$$

With equality if $a > 0$

3) (35 Points) Consider the following maximization problem faced by an economics graduate student:

$$\max \sum_{t=0}^{\infty} \beta^t \left[\gamma ln(l_t) + (1-\gamma) ln(1-n_t)\right]$$

Subject to:

$$l_t + k_{t+1} \leq k_t^{\alpha} n^{1-\alpha}$$

$$l_t, k_{t+1}, n_t \geq 0$$

$$k_0 \text{ given}$$

where l_t is learning, $(1 - n_t)$ represents sleep, and k_t is human capital.

(a) (15 Points) Write this problem recursively. Identify the state and choice variables.

Note that I am substituting the constraint in. You do not have to do this:

$$V(k) = \max_{k', n \geq 0} \gamma ln(k^{\alpha} n^{1-\alpha} - k') + (1-\gamma) ln(1-n) + \beta V(k')$$

The choice variables are k' and n, the state variable is k.

(b) (20 Points) Now, let z_t denote an exogenous productivity shock to the production function (i.e. we now have $k_t^\alpha n^{1-\alpha} z_t$). z_t follows the following distribution:

$$z_t = \begin{cases} z_l & \text{with prob. } \frac{1}{2} \\ z_h & \text{with prob. } \frac{1}{2} \end{cases}$$

Write this new problem recursively. Identify the state and choice variables.

$$V(k,z) = \max_{k',n \geq 0} \gamma ln(k^\alpha n^{1-\alpha} z - k') + (1-\gamma) ln(1-n) + \beta \mathbb{E}[V(k', z')]$$

The choice variables are k' and n, the state variables are k and z.

(c) (5 Points) **Extra Credit:** Solve the Bellman equation from part (a) using guess and verify with the guess V(k) = A + Bln(k). Find the policy function for human capital, as well as the optimal levels of l and n.

Taking our F.O.C with respect to k' and n:

$$\frac{\gamma}{k^\alpha n^{1-\alpha} - k'} = \frac{\beta B}{k'} \tag{1}$$

$$\frac{\gamma(1-\alpha)k^\alpha n^{-\alpha}}{k^\alpha n^{1-\alpha} - k'} = \frac{(1-\gamma)}{(1-n)} \tag{2}$$

Solving (1) for k' gives:

$$k' = \frac{\beta B k^\alpha n^{1-\alpha}}{\gamma + \beta B} \tag{3}$$

Plugging (3) into (2) gives:

$$n^* = \frac{(1-\alpha)(\gamma + B\beta)}{(1-\alpha)(\gamma + B\beta)(1-\gamma)} \tag{4}$$

Having found our optimal k' and n^*, we can plug this into the RHS of our value function:

$$RHS = \gamma ln\left(k^\alpha n^{*(1-\alpha)} - \frac{\beta B k^\alpha n^{1-\alpha}}{\gamma + \beta B}\right) + (1-\gamma) ln(1-n^*) + \beta\left(A + B ln\left(\frac{\beta B k^\alpha n^{1-\alpha}}{\gamma + \beta B}\right)\right)$$

After rearranging terms and setting the RHS equal to the LHS we get:

$$B = \alpha\gamma + \alpha\beta B$$

Which gives us:

$$B = \frac{\alpha\gamma}{1 - \alpha\beta}$$

and Consequently:

$$n^* = \frac{\gamma(1-\alpha)}{\gamma(1-\alpha) + (1-\gamma)(1-\alpha\beta)}$$

Now, using our solution for B and $*n^*$, we can show that:

$$\frac{1}{1-\beta}\left(\gamma ln(\gamma n^{*(1-\alpha)}) + \gamma\alpha ln(k) - \gamma ln(\gamma + \beta) + (1-\gamma) ln(1-n^*) + \beta\left(\frac{\alpha\gamma}{1-\alpha\beta}\right) ln(\beta n^* 1 - \alpha)\right)$$

Finally, using n^* and B we get that:

$$k^{'*} = \alpha\beta k^{\alpha} n^{*(1-\alpha)}$$

$$l^* = (1-\alpha\beta) k^{\alpha} n^{*(1-\alpha)}$$