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Midterm Exam
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This exam is closed book. Most points are given for the correct set-up of a problem and for economically insightful interpretations. You have 75 minutes for a maximum score of 100 points.

[30] **Problem 1.** *Refined Tastes.* Consider an infinite horizon Neoclassical Growth Model where consumers (indexed by a superscript i) maximize the present discounted value of lifetime utility over consumption. Consumers inelastically supply 1 unit of labor to earn income w_t^i and can borrow or save in any period t at some given rate of interest r_{t+1}. There is a restriction on assets, $a_t^i \geq -\overline{A}$, that is sufficiently negative such that it never binds.

Each period, the government levies equal lump-sum taxes T_t^i on each consumer to finance its government spending G_t. In particular, the government spends all of its tax revenues on fine Livermore Valley wine that it keeps for itself (in other words, the wine is never given to consumers).

[10] **(a)** Write down the consumer's problem *sequentially*. Identify the choice and state variables for some arbitrary period t.

State: $a_{t-1}^i, T_t^i, w_t^i, r_t$, Choice: c_t^i, a_t^i

$$\max_{\{c_t^i, a_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

$$\begin{aligned} \text{s.t.} \qquad & c_t^i + a_t^i \leq w_t^i + (1+r_t)a_{t-1}^i - T_t^i \quad \forall t \\ & a_t^i \geq -\overline{A} \quad \forall t \\ & c_t^i \geq 0 \quad \forall t \end{aligned}$$

[10] **(b)** Let depreciation be δ and suppose that the productive process in the economy can be summarized by a constant returns to scale production function $f(k_t)$, with k_0 given. Now, consider a scenario where the government decides how to allocate resources. It does so buy solving a Pareto problem.

In solving this problem, the government has determined that the optimal amount of wine to buy for itself each period is $\{G_t\}_{t=0}^{\infty}$, and that whatever is left should be split among the consumers with equal Pareto weights placed on everyone. Assuming a stand-in (i.e. representative) agent, write down the problem faced by the government when divvying up the resources. In addition, be sure to combine some constraints to produce the aggregate market clearing condition for the economy.

$$\max_{\{c_t, a_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \forall t$$

$$\begin{aligned} \text{s.t.} \qquad & c_t + a_{t+1} \leq f(k_t) - T_t \quad \forall t \\ & k_{t+1} = (1-\delta)k_t + a_t \quad \forall t \\ & T_t = G_t \quad \forall t \\ & a_t \geq -\overline{A} \quad \forall t \\ & c_t, k_t \geq 0 \quad \forall t \\ & k_0 \text{ given} \end{aligned}$$

To find the aggregate market clearing condition, we first note that $a_t = k_{t+1} - (1-\delta)k_t$ and $T_t = G_t$. Plugging both of these into the first constraint gives us

$$c_t + k_{t+1} + G_t \leq f(k_t) + (1-\delta)k_t \quad \forall t.$$

[10] **(c)** Returning back to part (a), suppose that optimizing consumers chose bundles summarized by $(\overline{c}^i)$ and profit maximizing firms produced $(\overline{y}^j)$, given prices. Prove that this allocation is Pareto Optimal.

Fairly detailed sketch:

We are told that, given prices, consumers are utility maximizing, firms are profit maximizing, and that the allocation was feasible (they chose it). This defines a *competitive equilibrium.* Next, summarize the price system by $\phi : S \to \mathbb{R}$, where S is some normed vector space that encompasses the choice sets of both consumers and firms.

Suppose that there exists another feasible allocation $[(\widetilde{c}^i), (\widetilde{y}^j), \phi]$ such that $u(\widetilde{c}^i) \geq u(\overline{c}^i)$ for all i, and strictly for at least one i. If so, it must be the case that

$$\phi\left(\sum_i \widetilde{c}^i\right) > \phi\left(\sum_i \overline{c}^i\right),$$

that is, the proposed allocation costs strictly more than the original allocation. Since ϕ is linear, this is only true if the total amount consumed in this alternate allocation is higher than in the original.

Since we also supposed that this alternate allocation was feasible, we'll similarly have that

$$\phi\left(\sum_j \widetilde{y}^j\right) > \phi\left(\sum_j \overline{y}^j\right).$$

The above states that profits under the alternate allocation are higher. However, we are given that firms are profit maximizing. The existence of an alternative (feasible) allocation with higher profits is a contradiction. That is, there cannot be another feasible allocation that makes everyone weakly better off (and at least one strictly so).

[40] **Problem 2.** *Memes are life, but time is money.* Consider an infinite horizon world where agents maximize (expected) utility over consumption and "likes" on facebook, $u(c, x)$, where x denotes the number of likes received in a period. Agents discount the future with a factor β and are endowed with 1 unit of time in each period, which can be spent writing a facebook post (denoted by t) or working and earning income $(1-t)w$.

"Likes" on facebook are not a guarantee, however. When spending t units of time writing a post, there is a probability $t\pi_s$ that your post is successful and delivers $\overline{x}$ likes. With the complement of that probability, $1 - t\pi_s$, your post is bad and you receive 0 likes. s denotes the current state of the internet economy and can be either high (h) or low (l), reflecting the relative ease with which "likes" may be obtained (where $\pi_h > \pi_l$). Let the state of the economy evolve over time as a two-state Markov process given by transition matrix $p(s'|s)$. That is,

$$x(t|s) = \begin{cases} \overline{x} & \text{with prob. } t\pi_s \\ 0 & \text{with prob. } 1 - t\pi_s \end{cases}, \qquad s \in \{h, l\} \text{ evolves according to } p(s'|s).$$

[10] **(a)** Assuming you can only post once per period, write down the problem of an agent in this economy *recursively*, being sure to clearly specify the state and choice variables.

State: s, Choice: c, t

$$V(s) = \max_{c,t} \left\{ \mathbb{E}\left[u(c,x)|s\right] + \beta \mathbb{E}\left[V(s')|s\right] \right\} \qquad \text{s.t.} \qquad c = (1-t)w$$

$$= \max_{t} \left\{ t\pi_s u\Big((1-t)w, \overline{x}\Big) + (1 - t\pi_s) u\Big((1-t)w, 0\Big) + \beta \sum_{s'} V(s') p(s'|s) \right\}$$

[10] **(b)** Suppose the utility function is given by $u(c, x) = ln(c) + \gamma x$, where $\gamma > 0$. Derive the policy function for an agent in this economy. Briefly explain what it says.

$$V(s) = \max_{t} \left\{ t\pi_s \Big[ln\big((1-t)w\big) + \gamma\overline{x}\Big] + (1 - t\pi_s)\Big[ln\big((1-t)w\big) + \gamma 0\Big] + \beta \sum_{s'} V(s') p(s'|s) \right\}$$

$$= \max_{t} \left\{ t\pi_s \gamma \overline{x} + ln\big((1-t)w\big) + \beta \sum_{s'} V(s') p(s'|s) \right\}$$

Now differentiate with respect to t (noting that the expectation over the future is not a function of t).

$$\frac{dV(s)}{dt} = \pi_s \gamma \overline{x} - \frac{1}{1-t} = 0 \qquad \Longrightarrow \qquad t^*(s) = 1 - \frac{1}{\pi_s \gamma \overline{x}}, \quad s \in \{h, l\}$$

The above says that, given some observed state s, the agent should spend the stated amount of time writing a facebook post, and the rest should be spent working.

[10] **(c)** Let's consider this world with reputation effects. In particular, suppose that the time spent writing a post in any period garners a following of fans who are guaranteed to like your post next period (but they only stick around for that one period). Now the "likes" one receives is given by

$$x(t|t^-, s) = \begin{cases} \overline{x} + \rho t^- & \text{with prob. } t\pi_s \\ \rho t^- & \text{with prob. } 1 - t\pi_s \end{cases},$$

where $\rho > 0$ and t^- denotes the time spent writing a post last period. Rewrite the agent's problem recursively, again clearly identifying any state and choice variables. (Maintain the functional form specification from the previous part, simplifying as much as possible.)

State: s, t^-, Choice: c, t

$$V(s, t^-) = \max_{c,t} \left\{ \mathbb{E}[u(c, x)|s, t^-] + \beta \mathbb{E}[V(s', t)|s] \right\} \qquad \text{s.t.} \qquad c = (1-t)w$$

$$= \max_t \left\{ t\pi_s u\Big((1-t)w, \overline{x} + \rho t^-\Big) + (1 - t\pi_s) u\Big((1-t)w, \rho t^-\Big) + \beta \sum_{s'} V(s', t) p(s'|s) \right\}$$

$$\vdots$$

$$= \max_t \left\{ t\pi_s \gamma \overline{x} + ln\big((1-t)w\big) + \gamma \rho t^- + \beta \sum_{s'} V(s', t) p(s'|s) \right\}$$

[10] **(d)** Continuing with part (c), suppose now that the state of the aggregate internet economy is constant over time (i.e. s never changes and we can write $\pi_s = \pi$). Solve for the agent's policy function and show that it doesn't actually depend on the state (in other words, we are finding the explicit solution for t every period).

First, simplify the problem we had from before, then take the FOC.

$$V(t^-) = \max_t \left\{ t\pi\gamma\overline{x} + ln\big((1-t)w\big) + \gamma\rho t^- + \beta V(t) \right\}$$

$$\frac{dV(t^-)}{dt} = \pi\gamma\overline{x} - \frac{1}{1-t} + \beta\frac{dV(t)}{dt} = 0$$

$$\implies \quad \beta\frac{dV(t)}{dt} = \frac{1}{1-t} - \pi\gamma\overline{x}$$

How do we find $dV(t)/dt$? Since, for optimizing agents, this relationship will be true every period, let's find $dV(t^-)/dt^-$ and then "push forward" one period.

$$\frac{dV(t^-)}{dt^-} = \gamma\rho \qquad \implies \qquad \frac{dV(t)}{dt} = \gamma\rho$$

Plug this into the FOC and solve for t.

$$t^* = 1 - \frac{1}{\gamma(\pi\overline{x} + \beta\rho)}$$

We can quickly verify that the above does not depend on t^-.

[30] **Problem 3.** *A Dynastic Economy.* Consider an economy with dynasties. A dynasty consists of an infinite sequence of agents. Each of the agents lives for one period, cares about the utility of her children, and makes decisions about her consumption c_t, number of children n_t, and next period assets a_{t+1}. To simplify matters, assume that the number of children n_t is a continuous (and non-negative) variable.

The representative dynasty (equivalently, the agent living in period zero) has preferences given by

$$U\left(\{c_t, n_t\}_{t=0}^{\infty}\right) = u(c_0) + \beta(n_0)u(c_1) + \beta(n_0)\beta(n_1)u(c_2) + \beta(n_0)\beta(n_1)\beta(n_2)u(c_3) + \dots.$$

The discount factor between period t and $t+1$, $\beta(n_t)$, thus depends on the number of children. It is assumed that $\beta(n)$ is increasing in n and that $\beta(n) < 1$ for all $n \geq 0$.

Raising children is time consuming. The time required to raise one child is exogenously given by x. The agents are endowed with one unit of time and spend the rest of their time $1 - xn_t$ by working. The rate of return on assets r and the wage rate w are both assumed exogenous and constant over time. The budget constraint is

$$c_t + a_{t+1} \leq w(1 - xn_t) + (1+r)a_t.$$

Next period assets, a_{t+1}, are equally divided among all children. Initial assets a_0 are given.

[5] **(a)** Write down the Bellman equation (BE) corresponding to the sequence problem described above.

$$V(a) = \max_{a',n,c} \left\{ u(c) + \beta(n) V\left(\frac{a'}{n}\right) \right\} \qquad \text{s.t.} \qquad c + a' \leq w(1 - xn) + (1+r)a$$

The questions that follow involve modifications to the BE in part (a). In each case, apply the indicated modification to the original BE. That is, do not answer (c) by modifying the equation you have written for (b).

[5] **(b)** Modify the BE so as to incorporate the assumption that the wage rate w is stochastic, and can take the value w_L with probability π, and the value w_H with the complement of that probability.

$$V(a, w) = \max_{a',n,c} \left\{ u(c) + \beta(n) \left[\pi V\left(\frac{a'}{n}, w_L\right) + (1-\pi) V\left(\frac{a'}{n}, w_H\right) \right] \right\}$$

$$\text{s.t.} \qquad c + a' \leq w(1 - xn) + (1+r)a$$

[10] **(c)** Modify the BE so as to incorporate the assumption that with probability $1 - q$ one half of the children will die before the end of the current period. Assume that the time costs of raising these children have been fully spent and that, if those children die, next period assets are divided equally among the surviving children.

$$V(a) = \max_{a',n,c} \left\{ u(c) + q\beta(n) V\left(\frac{a'}{n}\right) + (1-q)\beta\left(\frac{n}{2}\right) V\left(\frac{2a'}{n}\right) \right\}$$

$$\text{s.t.} \qquad c + a' \leq w(1 - xn) + (1+r)a$$

[10] **(d)** Modify the BE so as to incorporate the assumption that the time spent with each child x is chosen by the parents and affects human capital of the children. Specifically, if parents have human capital h_t then each child will have human capital $G(h, x)$, where G is increasing in both arguments. Human capital is useful because it increases earnings: an agent with human capital h earns wage wh.

$$V(a,h) = \max_{a',n,c,x} \left\{ u(c) + \beta(n) V\left(\frac{a'}{n}, h'\right) \right\}$$

$$\text{s.t.} \qquad c + a' \leq wh(1 - xn) + (1 + r)a$$
$$h' = G(h, x)$$