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Econ 204B

Midterm Question

Problem 2 *Equilibrium Firm Dynamics.* Consider an infinite time model of firm dynamics with firm entry and exit. Firms produce q_t units of a homogeneous consumption good that they sell in a perfectly competitive market at price p_t. Operating firms incur a fixed cost, c_f, and a variable cost defined by $c_v^i = \frac{1}{\psi_i} w_t q^2$, where ψ_i is firm $i's$ productivity and w_t is the wage rate faced by all firms. Moreover, assume that an incumbent's productivity evolves according to $\psi_{t+1}^i = \rho\psi_t^i + \epsilon_{t+1}^i$, where $\epsilon_t^i \sim \mathcal{N}(0, \sigma_\psi^2)$ and $0 < \rho < 1$. New firms may enter the market at which point they become incumbants and receive an i.i.d. draw from $L\mathcal{N}(0, \sigma_\psi^2)$. At the start of each period, incumbents may exit. Only firms who are active incur the fixed cost of producing and firms who exit the market are assumed to be destroyed. Assume that p_t and w_t are exogenous and non-stochastic, and that firms discount future profits at rate β.

a) Write the problem of an incumbent and potential entrant at the start of each period. Be sure to identify the choice and state variable(s).

The value function for an incumbent is

$$V(\psi, p, w) = \max\{0, V_f(\psi, p, w)\}$$

where,

$$V_f(\psi, p, w) = \max_{q \geq 0} p \cdot q - \frac{1}{\psi} w q^2 - c_f + \beta \mathbb{E}(V(\psi', p', w')|\psi)$$

An entrant faces the following problem:

$$W = \max\{0, \mathbb{E}V(\psi, p, w)\}$$

b) (10 pts.) Now suppose that an incumbent's productivity evolves according to $\psi_{t+1}^i = \epsilon_{t+1}^i$, where $\epsilon_t^i \sim L\mathcal{N}(0, \sigma_\psi^2)$. That is, assume that ψ^i is i.i.d. In addition, assume that there is a one time fixed cost to entry c_e. Does your answer to part (a) change? If so, re-write the problem for both types of firms and explain what has changed. If nothing changes, explain why not.
An encumbant faces the following problem:

$$V(\psi, p, w) = \max\{0, V_f(\psi, p, w)\}$$

where

$$V_f(\psi, p, w) = \max_{q \geq 0} p \cdot q - \frac{1}{\psi} w q^2 - c_f + \beta \mathbb{E}V(\psi', p', w')$$

An entrant faces the following problem:

$$W = \max\{0, \mathbb{E}V(\psi, p, w) - c_e\}$$

c) Maintaining the assumptions from part b, completely characterize(i.e. solve for) and explain the strategy for both types of firms in equilibrium.

The incumbent's value function is given by

$$V(\psi, p, w) = \max\{0, V_f(\psi, p, w)\}$$

where

$$V_f(\psi, p, w) = \max_{q \geq 0} p \cdot q - \frac{1}{\psi} w q^2 - c_f + \beta \mathbb{E}(V(\psi', p', w'))$$

Notice that, conditional on producing, the optimal strategy for the firm is a simple profit maximization

$$q^* = \frac{p\psi}{2w}$$

$$\Rightarrow V_f(\psi, p, w) = \frac{p^2\psi}{2w} - \frac{p^2\psi}{4w} - c_f + \beta \mathbb{E} V(\psi', p', w')$$

Moreover, firms stay if $\psi > \psi^*$ and exit otherwise, where

$$\frac{p^2\psi^*}{2w} - \frac{p^2\psi^*}{4w} - c_f + \beta \mathbb{E}(V(\psi', p', w')) = 0$$

The entrant's value function is given by

$$W(p, w) = \max\{0, \mathbb{E}(V(\psi, p, w)) - c_e\}$$

Moreover, an entrant will choose to enter so long as $c_e < \mathbb{E}(V(\psi, p, w))$. Thus, firms will enter until $c_e = \mathbb{E}(V(\psi, p, w))$

$$\Rightarrow \psi^* = \frac{4w \cdot (c_f - c_e)}{p^2}$$

Problem 1 *Human Capital.* Consider an economy where there is a measure one of infinitely lived agents. Preferences are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where c_t is consumption in period t. Agents can accumulate human capital h_t. All agents have the same initial human capital stock, h_0. Let H_t denote the average human capital in the population in period t. Human capital can be used either to produce output, or to increase future human capital. Let n_t be the part of human capital allocated to market activities (individual's labor input) and $h_t - n_t$ be the part allocated to enhancing human capital. Assume that human capital evolves according to

$$h_{t+1} = h_t + \sigma(h_t - n_t)^{\frac{1}{2}}$$

People are not allowed to borrow or save and there is no physical capital in the economy. The production technology is thus given by

$$Y_t = z_t N_t H_t^{\gamma} \qquad\qquad 0 < \gamma < 1$$

where Y_t is output, N_t is labor input, and z_t is a finite state first order Markov process that takes values from $Z = \{z_1, z_2, \ldots, z_n\}$ with all values being strictly positive.

a) (15 pts.) Write down the household's problem recursively. Be explicit about which variables are choice variables, which are state variables, and any constraints faced by households. Does the household's problem change if z where a second order Markov process? If so, show how and explain why. If not, exlpain why not.

$$V(h, H, z|G) = \max_{h', c, n}\{u(c) + \beta \sum_{z'} V(h', z', H'|G)\pi(z'|z)\}$$

$$s.t. \qquad c \leq w(H)n$$

$$c, h', n \geq 0$$

$$c = \sigma^c(z, h, H|G)$$

$$h' = \sigma^h(z, h, H|G) = h + \sigma(h - n)^{\frac{1}{2}}$$

$$n = \sigma^n(z, h, H|G)$$

$$H' = G(z, H)$$

If instead z were a second order Markov process, then z_{-1} would also be a state variable. Moreover, the value function and decision rules, σ would all be a function of z_{-1} as well. For example, the value function would instead be

$$V(h, H, z, z_{-1}|G) = \max_{h', c, n}\{u(c) + \beta \sum_{z'} V(h', z', H'|G)\pi(z'|z, z_{-1})\}$$

b) (15 pts.) Define the rational expectations recursive competitive equilibrium for the economy. A rational expectations recursive competitive equilibriym is a value function, $V(h, z, H|G)$, policy functions, $\sigma^i(h, z, H|G)$ where $i \in \{c, n, h'\}$, a pricing function, $w(z, H)$, and an aggregate law of motion, $G(z, H)$, such that

1) The value fcuntion and policy functions solve the household's problem taking prices and the aggregate law of motion as given.

2) Firm's maximize profits so that $w(z, H) = zH^\gamma$.

3) The aggregate law of motion is consistent $\sigma^H(z, H, H|G) = G(z, H)$.

4) Markets clear so that $\sigma^c(z, H, H|G) = zNH^\gamma$.

c) (10 pts.) Write down the social planner's problem recursively. Be explicit about which variables are choice variables, which are state variables, and any constraints that the social planner faces. Has anything changed from part (a)? If so, show how and explain why. If not, explain why not.

$$V(z, H) = \max_{C,N,H'} \{u(c) + \beta \sum_{z'} V(H', z')\pi(z'|z)\}$$

$$s.t. \qquad C \leq zNH^\gamma$$

$$N, C, H' \geq 0$$

$$H' = H + \sigma(H - N)^{\frac{1}{2}}$$

Yes. Here, only H and z are state variables. This is because the social planner "controls" the entire economy and thus only needs to keep track of aggregate variables. Said differently, the social planner does not need to incorporate prices into their decision problem. Additionally, the social planner observes the aggregate law of motion and thus does not need to condition their choices on their expectation of said law of motion. Lastly, the budget constraint is now a resource constraint.

d) (20 pts.) Show explicitly whether or not the Second Welfare Theorem holds. That is, show explicitly whether or not the solution to the social planner's problem and household's problem are the same. If they are the same, explain why. If they are not the same, explain why not.

Planner's Problem:

$$\mathcal{L} = u(zNH^{\gamma}) + \beta \sum_{z'} V(z', H')\pi(z'|z) + \lambda(H + \sigma(H - N)^{\frac{1}{2}}) - H')$$

$$\Rightarrow \mathcal{L}_N = zH^{\gamma}u'(zNH^{\gamma}) + \beta \sum_{z'} \tfrac{\partial V(z',H')}{\partial N}\pi(z'|z) - \lambda(\tfrac{\sigma}{2}(H - N)^{\frac{-1}{2}}) = 0$$

$$\Rightarrow \mathcal{L}_{H'} = \beta \sum_{z'} \tfrac{\partial V(z',H')}{\partial H'}\pi(z'|z) - \lambda = 0$$

$$\Rightarrow \frac{\mathrm{d}V(z,H)}{\mathrm{d}H} = zN\gamma H^{\gamma-1}u'(zNH^{\gamma}) + \lambda(1 + \frac{\sigma}{2}(H - N)^{\frac{1}{2}})$$

Household's Problem:

$$\mathcal{L} = u(w(z,H)n) + \beta \sum_{z'} V(z', h', H'|G)\pi(z'|z) + \lambda(h + \sigma(h - n)^{\frac{1}{2}})$$

$$\Rightarrow \mathcal{L}_n = w(z,H)u'(w(z,H)n) - \lambda(\frac{\sigma}{2}(h - n)^{\frac{-1}{2}}) = 0$$

$$\Rightarrow \mathcal{L}_{h'} = \beta \sum_{z'} \tfrac{\partial V(z',h',H'|G)}{\partial h'}\pi(z'|z) - \lambda = 0$$

$$\Rightarrow \frac{\mathrm{d}V(z,h,H|G)}{\mathrm{d}h} = \frac{\partial u(w(H)n)}{\partial h} + \lambda(1 + \frac{\sigma}{2}(h - n)^{\frac{-1}{2}})$$

If we endow one household will all of the capital and all of the labor in the economy, all of the first order conditions for the household are identical to that of the social planner except for the envelope condition that becomes

$$\frac{\mathrm{d}V(z,H,H|G)}{\mathrm{d}h} = \lambda(1 + \frac{\sigma}{2}(H - N)^{\frac{-1}{2}})$$

Thus, the social planner and household problems are not equivalent. The reason being that there is an externality of human capital accumulation because $\gamma > 0$ that individual households don't internalize. This important feature of the model causes the shadow value of human capital, and thus its stock, to be too low in the competitive equilibrium.