ECON 204B Final

1) Consider the problem of an individual who is trying to find as spouse. They are impatient to find their match, i.e. they discount the future at rate $r > 0$. They know that they will encounter potential partners at random with some arrival rate α_0, and that the quality of the match is drawn from an exogenously given "attraction" distribution, $F(A), A \in [\underline{A}, \bar{A}]$. When unmarried, this individual receives bachelor utility, denoted $b < \underline{A}$, and when married they receive attraction utility A. Suppose that marriages end with some exogenously given divorce rate δ.

   a Derive the continuous time Bellman equations associated with being married and with being single. Denote them M(A) and S respectively.

$$M(A) = Adt + \frac{1}{1+rdt}\Big[(1-\delta dt)M(A) + \delta dt S\Big]$$

$$S = bdt + \frac{\alpha_0 dt}{1+rdt}\int_{\underline{A}}^{\overline{A}} \max\{M(A), S\}\, dF(A) + \frac{1-\alpha_0 dt}{1+rdt}S.$$

We can simplify S a little bit with some algebra.

$$\frac{rdt}{1+rdt}S = bdt + \frac{\alpha_0 dt}{1+rdt}\int_{\underline{A}}^{\overline{A}} \max\{M(A), S\}\, dF(A) - \frac{\alpha_0 dt}{1+rdt}S$$

$$\frac{rdt}{1+rdt}S = bdt + \frac{\alpha_0 dt}{1+rdt}\int_{\underline{A}}^{\overline{A}} \max\{M(A) - S, 0\}\, dF(A)$$

$$S = \frac{(1+rdt)b}{r} + \frac{\alpha_0}{r}\int_{\underline{A}}^{\overline{A}} \max\{M(A) - S, 0\}\, dF(A)$$

Now for M(A):

$$(1+rdt)M(A) = (1+rdt)Adt + (1-\delta dt)M(A) + \delta dt S \tag{1}$$

$$\tag{2}$$

$$rdtM(A) = (1+rdt)Adt + \delta dt\Big[S - M(A)\Big] \tag{3}$$

$$\tag{4}$$

$$M(A) = \frac{(1+rdt)A}{r} + \frac{\delta}{r}\Big[S - M(A)\Big] \tag{5}$$

Taking the limit as dt $\to 0$

$$M(A) = \frac{A}{r} + \frac{\delta}{r}\Big[S - M(A)\Big]$$
$$S = \frac{b}{r} + \frac{\alpha_0}{r}\int_{\underline{A}}^{\overline{A}} \max\{M(A) - S, 0\}\, dF(A)$$

Now, note that we can simplify M(A) further:

$$M(A) = \frac{A + \delta S}{r + \delta}$$

b Solve for the reservation level of attraction for which the individual will accept a match. First let's rewrite, rS. Recall that we are assuming $(M(A_r) = S)$

$$\begin{aligned}
rS &= b + \alpha_0 \int_{\underline{A}}^{\overline{A}} \max\{M(A) - S, 0\}\, dF(A) \\
&= b + \alpha_0 \int_{\underline{A}}^{\overline{A}} \max\Big\{\underbrace{\frac{A + \delta S}{r+\delta}}_{M(A)} - \underbrace{\frac{A_r + \delta S}{r+\delta}}_{S}, 0\Big\} dF(A) \\
&= b + \frac{\alpha}{r+\delta}\int_{A_R}^{\overline{A}} (A - A_R) dF(w) \\
&= b + \frac{\alpha}{r+\delta}\int_{A_R}^{\overline{A}} [1 - F(w)] dw && \text{(int. by parts)}
\end{aligned}$$

Which is our reservation A.

c Now, suppose we institute "On-the-job search", meaning that after matching an individual can keep searching for a new partner with arrival rate $\alpha_1 < \alpha_0$. Find the new Bellman equations. Solve for the reservation wage.

First, note that this is identical to what we saw in section (with different notation), thus refer to those slides for a slightly more detailed breakdown of the algebra. We begin by setting up our new married Bellman (the single Bellman remains the same:

$$
\begin{aligned}
M(A) = Adt + \frac{1}{1+rdt}\Bigg[& \alpha_1 dt(1-\delta dt)\int_{\underline{A}}^{\overline{A}} \max\left\{V(\widetilde{A}), M(A)\right\} dF(\widetilde{A}) \\
& + (1-\alpha_1 dt)(1-\delta dt)M(A) \\
& + \alpha_1 \delta dt^2 \int_{\underline{A}}^{\overline{A}} \max\left\{M(\widetilde{A}), S\right\} dF(\widetilde{A}) \\
& + (1-\alpha_1 dt)sdtS \Bigg]
\end{aligned}
$$

After simplifying and setting $dt \to 0$, we get:

$$
rM(A) = A + \alpha_1 \int_{\underline{A}}^{\overline{A}} \max\left\{M(\widetilde{A}) - M(A), 0\right\} dF(\widetilde{A}) + \delta\Big[S - M(A)\Big]
$$

To find the reservation strategy we begin by finding $rM(A_R)$:

$$
\begin{aligned}
rM(A_R) &= A_R + \alpha_1 \int_{\underline{A}}^{\overline{A}} \max\left\{M(\widetilde{A}) - M(A_R), 0\right\} dF(\widetilde{A}) + \delta\Big[\underbrace{S - M(A_R)}_{=0}\Big] \\
&= A_R + \alpha_1 \int_{A_R}^{\overline{A}} [M(\widetilde{A}) - M(A_R)] dF(\widetilde{A})
\end{aligned}
$$

Applying integration by parts:

$$
rM(A_R) = A_R + \alpha_1 \int_{A_R}^{\overline{A}} M'(\widetilde{A})[1 - F(\widetilde{A})] d\widetilde{A}
$$

Next, we construct $rM(A_R) = S$ and solve for A_R:

$$
\begin{aligned}
A_R + \alpha_1 \int_{A_R}^{\overline{A}} V'(\widetilde{A})[1 - F(\widetilde{A})] d\widetilde{A} &= b + \alpha_0 \int_{A_R}^{\overline{A}} [M(\widetilde{A}) - S] dF(\widetilde{A}) \\
&= b + \alpha_0 \int_{A_R}^{\overline{A}} V'(\widetilde{w})[1 - F(\widetilde{A})] d\widetilde{A}
\end{aligned}
$$

$$
A_R = b + (\alpha_0 - \alpha_1) \int_{A_R}^{\overline{A}} M'(\widetilde{A})[1 - F(\widetilde{A})] d\widetilde{A}
$$

Finally, solve for $M'(A)$ and plug it in:

$$M'(A) = \frac{1}{r + \delta + \alpha_1[1 - F(A)]}$$

So:

$$A_R = b + (\alpha_0 - \alpha_1) \int_{A_R}^{\overline{A}} \frac{[1 - F(A)]}{r + \delta + \alpha_1[1 - F(A)]} dA$$

d Now, suppose that separating from your spouse in favor of a new one has some fixed cost λ. How does this change the reservation strategy from part (c)?

It will increase your reservation level of attractiveness when you are Single, and make it so that you do not accept any new offer such that $\tilde{A} > A$. The new partner's A must counterbalance the cost of divorce.

e Now, suppose that $\alpha_1 > \alpha_0$. What is the new reservation strategy? Hint: You do not need to solve for it.

The reservation strategy is to accept any offer. The instantaneous utility from being single is less than the lowest offer ($b < \bar{A}$) and you receive new offers at a faster rate when married so there is no incentive to stay single.

2) Consider a continuous time search model with crime. Agents discount the future at rate $r > 0$ and receive unemployment utility flow b. Firms discount future profits at rate r, and pay a flow cost $\gamma > 0$ to advertise a vacancy. Vacant firms produce no output while filled jobs produce output $y > b$.

The flow of contacts between firms and workers follows a matching function m(u,v). Let $\theta = \frac{v}{u}$ denote labor market tightness. Vacancies are filled with arrival rate $\frac{m(u,v)}{v} = \alpha$ and unemployed workers find jobs with arrival rate $\frac{m(u,v)}{u} = \tilde{\alpha}$. When employed workers receive constant wage w and separate from their job with exogenous arrival rate s. Wages are determined by Nash Bargaining.

Finally, suppose that individuals receive an opportunity to commit crimes with arrival rate λ_i, where i indicates the individuals state (i = u for unemployed, i = e for employed). The value of a crime is ϵm, where $m \geq 0$ and ϵ is a random draw from a distribution $G(\epsilon)$ with support $[0, \bar{\epsilon}]$. An agent who commits a crime is caught and sent to jail with probability π. When in jail individuals receive flow utility x, and exit jail with arrival rate δ.

a Find the individual's flow bellman equations.

$$rV_u = b + \tilde{\alpha}(V_e - V_u) + \lambda_u \int \max\{\epsilon m + \pi(V_p - V_u), 0\} dG(\epsilon) \tag{1}$$

$$rV_e = w + s(V_u - V_e) + \lambda_e \int \max\{\epsilon m + \pi(V_p - V_u), 0\} dG(\epsilon) \tag{2}$$

$$rV_p = x + \delta(V_u - V_p) \tag{3}$$

b Find the firm's flow bellman equations.

Let ϵ_u and ϵ_e denote your reservation crime value when employed and unemployed respectively:

$$rV_v = -\gamma + \alpha(V_f - V_v) \tag{4}$$

$$rV_f = y - w - s(V_f - V_v) - \lambda_e \pi[1 - G(\epsilon_e)](V_f - V_v) \tag{5}$$

c Find the wage resulting from the Nash Bargaining Process. Note: You may assume free-entry for firms.

$$w = \text{argmax}\,(V_e - V_u)^\beta (V_f - V_v)^{1-\beta}$$

Assuming free-entry, we can set $V_v = 0$, thus, taking our FOC and simplifying we get:

$$(1-\beta)(V_e - V_u) = \beta V_f \tag{6}$$

What's more, note that (4) tells us:

$$V_f = \frac{\gamma}{\alpha}$$

Thus we can rewrite (6):

$$(V_e - V_u) = \frac{\beta\gamma}{(1-\beta)\alpha} \tag{7}$$

Note (1) and (2) imply that the employee's surplus is given by:

$$[r + s + \tilde{\alpha}](V_e - V_u) = w - b + \lambda_e m \int_{\epsilon_e}^{\bar{\epsilon}} [1 - G(\epsilon)]d\epsilon - \lambda_u m \int_{\epsilon_u}^{\bar{\epsilon}} [1 - G(\epsilon)]d\epsilon \tag{8}$$

Thus, combining (7) and (8) and solving the resulting equation for w gives us:

$$w = [r + s + \tilde{\alpha}] \left(\frac{\beta\gamma}{(1-\beta)\alpha} \right) + b - \lambda_e m \int_{\epsilon_e}^{\bar{\epsilon}} [1 - G(\epsilon)]d\epsilon + \int_{\epsilon_u}^{\bar{\epsilon}} [1 - G(\epsilon)]d\epsilon$$

Which is our wage.

d Now, suppose that we introduce stigma into the market. Namely, after a person spends time in prison the arrival rate of offers changes to $\tilde{\alpha}_c < \tilde{\alpha}$. Find the individual's flow bellman equations. What effect will this have on their eventual wage?

Note there are multiple ways to set this up. This is one of them. Let C be an indicator variable which is equal to 1 if you have previously been convicted of a crime.

$$rV_u = b + (1 - C)\tilde{\alpha}(V_e - V_u) + C\tilde{\alpha}_c(V_e - V_u) + \lambda_u \int \max\{\epsilon m + \pi(V_p - V_u), 0\} dG(\epsilon)$$

$$rV_e = w + s(V_u - V_e) + \lambda_e \int \max\{\epsilon m + \pi(V_p - V_u), 0\} dG(\epsilon)$$

$$rV_p = x + \delta(V_u - V_p)$$

The stigma will lower their wage as it reduces their threat point (the value of being unemployed).