# Problem Set 2

**Problem 1.** *A model with search intensity.* Consider a search economy in which wages are given by an exogenous distribution $F(w)$ with a support bounded by $\underline{w}$ and $\overline{w}$. Matches are determined by an exogenous matching rate $\lambda(e)$, where $e \geq 0$ is their search effort while unemployed. $\lambda(e)$ is assumed to be strictly increasing and concave with $\lambda'(0) \to \infty$, and individuals face strictly increasing and convex search costs $c(e)$ with $c'(0) = 0$. During unemployment, agents receive flow utility $b$, and during employment they consume their wage. Matches end exogenously at a rate $s$.

**(a)** Write the flow Bellman equations for the employment and unemployment states.

$$rV(w) = w + s\Big[U - V(w)\Big]$$

$$rU = \max_e \left\{ b - c(e) + \lambda(e) \int_{\underline{w}}^{\overline{w}} \max\Big[V(w) - U, 0\Big] dF(w) \right\}$$

**(b)** Letting $e^*$ denote the optimal search intensity, derive the reservation wage. Be sure to solve as extensively as possible.

The reservation wage is defined by $rV(w_R) = rU$. First, construct $rV(w_R)$.

$$rV(w_R) = w_R + s\Big[U - V(w_R)\Big] \qquad \Longrightarrow \qquad rV(w_R) = r\frac{w_R + sU}{r+s}.$$

Since we know this equals $rU$, we can determine that

$$r\frac{w_R + sU}{r+s} = rU \qquad \Longrightarrow \qquad w_R = rU.$$

Next, note that we can rewrite the flow value of unemployment as follows.

$$rU = b - c(e^*) + \lambda(e^*) \int_{w_R}^{\overline{w}} \Big[V(w) - U\Big] dF(w)$$

Further, we can determine that

$$V(w) - U = \frac{w + sU}{r+s} - \frac{w_R + sU}{r+s} = \frac{w - w_R}{r+s}.$$

We can use the above to solve for the reservation wage.

$$
\begin{aligned}
w_R &= b - c(e^*) + \lambda(e^*) \int_{w_R}^{\overline{w}} \Big[V(w) - U\Big] dF(w) \\
&= b - c(e^*) + \frac{\lambda(e^*)}{r+s} \int_{w_R}^{\overline{w}} \Big[w - w_R\Big] dF(w) \\
&= b - c(e^*) + \frac{\lambda(e^*)}{r+s} \int_{w_R}^{\overline{w}} [1 - F(w)] dw \qquad \text{(integration-by-parts)}
\end{aligned}
$$

**(c)** Solve for the optimal search intensity, $e^*$. In doing so, you should show that the optimal search intensity maximizes the reservation wage (which is a function of $e$).

Rewrite the flow value of unemployment. Note that I am explicitly writing the reservation wage as a function of $e$. The optimal search intensity will then pin down the "optimal" reservation wage.

$$
rU = \max_e \left\{ b - c(e) + \frac{\lambda(e)}{r+s} \underbrace{\int_{w_R(e)}^{\overline{w}} [1 - F(w)] dw}_{\equiv \varphi(w_R(e))} \right\}
$$

Now, differentiate w.r.t. $e$ and set equal to 0 (let primes denote derivatives).

$$
-c'(e^*) + \frac{\lambda'(e^*)}{r+s} \varphi(w_R(e^*)) + \frac{\lambda(e^*)}{r+s} \varphi'(w_R(e^*)) w_R'(e^*) = 0
$$

We can straightforwardly determine what $\varphi'(w_R(e))$ is and use our earlier results to figure out what $w_R'(e)$ is.

$$
\varphi'(w_R(e)) = -[1 - F(w_R(e))]
$$

$$
w_R'(e) = -c'(e) + \frac{\lambda'(e)}{r+s} \varphi(w_R(e)) + \frac{\lambda(e)}{r+s} \varphi'(w_R(e)) w_R'(e).
$$

Interestingly, we can see that the last statement should be equal to 0 (it's the same as the FOC).[1] That is, the optimal choice of $e$ will maximize the reservation wage ($w_R'(e^*) = 0$). Plugging these results in to the FOC, we are left with an expression that pins down the optimal search intensity, $e^*$.

$$c'(e^*) = \frac{\lambda'(e^*)}{r+s}\varphi(w_R(e^*))$$

We can quickly verify that there will be a unique $e^*$ that solves the above given the assumptions on $\lambda(\cdot)$ and $c(\cdot)$.

**Problem 2.** Consider a model economy with continuum of individuals where each individual faces idiosyncratic labor productivity shocks. Each individual solves the following program

$$\max_{c,a'} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + a_{t+1} = r \cdot a_t + wexp(e)$$

where $c$ is consumption, $a$ is asset holdings, $r$ is the rate of return on assets, $w$ is the wage rate per efficiency units, $e$ is labor productivity. Individual labor productivity is governed by an AR(1)

$$e_{t+1} = \rho e_t + \epsilon_{t+1} \qquad \epsilon_t \sim N(o, \sigma^2)$$

The utility function can be parameterized as

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

Using the Tauchen-Hussey algorithm we get the following two-stage discrete Markov approximation of the estimated AR(1) process for individual productivity

$$\begin{bmatrix} e_h \\ e_l \end{bmatrix} = \begin{bmatrix} 0.1 \\ \ln(2 - exp(e_h)) \end{bmatrix}$$

and

$$\Pi = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

You should use piecewise linear splines for approximation and golden-section search for optimization when solving the household's problem. Use Monte Carlo methods to compute the distribution functions for assets and consumption implied by the individuals' optimal decision rules. Here, assume that $\gamma = 2$, $\beta = 0.98$, and $(r, w) = (1.0204, 1.9521)$. Set the asset grid to span from 0 to 200.

**(a)** Write code which performs the necessary value function iteration in this environment. Plot the policy and value functions.

See code on my website.

[1] One might also note that because it must be that $rU = w_R(e)$, maximizing $rU$ w.r.t. $e$ necessitates maximizing $w_R(e)$ w.r.t. $e$.