Econ 204C
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Problem set 1: Extraction and experimentation

The following questions use basic recursive representations in problems that are not conventionally seen in the realm of macroeconomics. The problems provide good practice to get familiar with dynamic programing. If you have any issues with the homework please contact me. Good luck!

Problem 1. Consider a planner endowed with a finite amount of a non-renewable resource, say oil. The initial value of this resource is x_0 and time is discounted at $\beta < 1$. Extracting oil is costly. Let y denote the amount extracted during a period and consider a cost of extraction of the form $C(x, y) = y^2 - yx$. Notice that $C_y(y, x) > 0$ (so long $y < x/2$) and that $C_{yy}(y, x) > 0$ so its increasingly costly to extract oil, but that $C_{xy}(y, x) < 0$ so the marginal cost of extraction is lower if the reserve is higher. [If you're curious, the previous set up is an example of more general problems associated with a *Linear-Quadratic Regulator*.]

The sequential problem is

$$v^*(x_0) \equiv \max_{\{y_t\}_{t=0}^{\infty}} \sum\nolimits_{t=0}^{\infty} \beta^t (x_t - y_t) y_t, \text{ s.t., } x_{t+1} = x_t - y_t, \text{ with } x_0 \text{ given.}$$

The objective function is simply $-C(x, y)$ as we are dealing with maximization (not cost minimization).

(a) Write the Bellman equation that represents the decision problem of the planner.

$$V(x) = \max_{0 \le y \le x} \{(x - y)y + \beta V(x - y)\}$$

(b) Conjecture (incorrectly, by the way) that the value function is linear, as in $v(x) = Ax$. Show that your conjecture is incorrect.

Suppose that V(x) = Ax for some $A > 0$. Then:

$$Kx = \max_{0 \le y \le x} \{(x - y)y + \beta A(x - y)\} \quad (1)$$
$$= \max_{0 \le y \le x} \{(x - \beta A)y - y^2 + \beta Ax\} \quad (2)$$

We know the optimal y(x) can be found by Taking a derivative with respect to y:

$$y(x) = \frac{(x - \beta A)}{2}$$

Plugging this in to our value function and simplifying gives us:

$$Ax = \frac{(x - \beta A)^2}{4} + \beta Ax$$

Which is clearly quadratic, not linear. Thus our conjecture is incorrect and V(x) cannot be linear.

(c) Suppose that $\beta = 0$ and solve for $v^*(x_0)$ or $v(x)$. (That is, find the sequential or the recursive value function, which should be the same as there are no dynamics.)

When $\beta = 0$ we are just maximizing the function $(x-y)y$. Doing this gives us:

$$y(x) = \frac{x}{2}$$

Thus:

$$v(x) = (x - \frac{x}{2})(\frac{x}{2}) = \frac{x^2}{4}$$

(d) The previous value function should give you a valid conjecture for the value function. Find the value function and the policy function associated with the extraction problem.

Based on (c), our conjecture will be that $v(x) = Ax^2$. Plugging this into the RHS of the Bellman:

$$\max_{0 \leq y \leq x} (x-y)y + \beta A(x-y)^2$$

Taking a derivative and setting equal to 0 gives us:

$$x - 2y - 2\beta A(x-y) = 0$$

and consequently:

$$y(x) = \left[\frac{1-2\beta A}{2(1-\beta A)}\right] x$$

Plugging this into our Bellman:

$$Ax^2 = \left(x - \left[\frac{1-2\beta A}{2(1-\beta A)}\right] x\right) \left[\frac{1-2\beta A}{2(1-\beta A)}\right] x + \beta A \left(x - \left[\frac{1-2\beta A}{2(1-\beta A)}\right] x\right)^2$$

After a lot of algebra we can find that:

$$A = \frac{1}{4(1-\beta A)}$$

The solution(s) to which is:

$$A = \frac{4 \pm \sqrt{16 - 16\beta}}{8\beta}$$

Note, however, that only one of these solutions gives us a positive y, thus, after simplifying, we find:

$$V(x) = \frac{1 - \sqrt{1-\beta}}{2\beta} x^2$$

and:

$$y(x) = \frac{x\sqrt{1-\beta}}{1 + \sqrt{1-\beta}}$$

Problem 2. Study Section 5.16 in SLP [Stokey, Lucas and Prescott] and solve Exercise 5.16. Since we haven't seen much of the technical material (although I'm sure you've cover some of it in 204B), thread lightly in part (a).

Note: See SLP for the problem setup.

(a) Show that $T : C \to C$, and that T satisfies Blackwell's sufficient conditions for a contraction. Hence T has a unique fixed point $v \in C$, Use Corollary 1 to the Contraction Mapping Theorem to show that v is HOD1.

First, we need to show that our operator

$$Tf(A, B) = \sup_{A \le P \le B} \left\{ \frac{B-P}{B-A}P + \beta \left[\frac{P-A}{B-A} f(A, P) + \frac{B-P}{B-A} f(P, B) \right] \right\}$$

maps a continuous function to another continuous function. To do this we will appeal to the maximum theorem. Here we know that the correspondence is a closed interval $[A, B] \in [0, 1]$, and thus a compact valued continuous correspondence, so its assumptions are satisfied. Thus we know $T : C \to C$.

Now, to show that T is a contraction we will check if it satisfies Blackwell's sufficiency conditions:

1. Monotonicity: Take f and $g \in C$ where $f(x) < g(x)$ for all x. We want to check if $Tf(x) \le Tg(x)$. Note that:

$$\frac{B-P}{B-A} f(P, B) \le \frac{B-P}{B-A} g(P, B)$$

and

$$\frac{P-A}{B-A} f(P, B) \le \frac{P-A}{B-A} g(P, B)$$

thus

$$Tf(A, B) \le Tg(A, B)$$

2. Discounting: Want to show that $T(f + a)(A, B) \le Tf(A, B) + \beta a$.

$$T(f+a)(A, B) = \sup_{A \le P \le B} \left\{ \frac{B-P}{B-A}P + \beta \left[\frac{P-A}{B-A} (f(A, P) + a) + \frac{B-P}{B-A} (f(P, B) + a) \right] \right\}$$

$$Tf(A, B) + \beta a = \sup_{A \le P \le B} \left\{ \frac{B-P}{B-A}P + \beta \left[\frac{P-A}{B-A} (f(A, P) + \frac{B-P}{B-A} f(P, B) + a \right] \right\}$$

so clearly $T(f + a)(A, B) = Tf(A, B) + \beta a$

Thus we can see the Blackwell's sufficient conditions are satisfied.

Finally, we want to show that v is HOD1. Note that Corollary 1 to the Contraction Mapping Theorem tells us that if T maps HOD1 functions into HOD1 functions, then the fixed point must be an HOD1 function. To see that our operator T does this consider the following. Let P^* be the solution to the original problem so that:

$$Tf(A, B) = \left\{ \frac{B-P^*}{B-A}P^* + \beta \left[\frac{P^*-A}{B-A} f(A, P^*) + \frac{B-P^*}{B-A} f(P^*, B) \right] \right\}$$

Now, scaling the original problem down we get:

$$Tf(\lambda A, \lambda B) = \left\{ \frac{\lambda B-P}{\lambda B-\lambda A}P + \beta \left[\frac{P-\lambda A}{\lambda B-\lambda A} f(\lambda A, P) + \frac{\lambda B-P}{\lambda B-\lambda A} f(P, \lambda B) \right] \right\}$$

Suppose now that we choose $P = \lambda P^*$ as a strategy and plug it in. We get:

$$\left\{ \frac{\lambda B - \lambda P}{\lambda B - \lambda A} \lambda P + \beta \left[\frac{\lambda P - \lambda A}{\lambda B - \lambda A} f(\lambda A, \lambda P) + \frac{\lambda B - \lambda P}{\lambda B - \lambda A} f(\lambda P, \lambda B) \right] \right\} = \lambda T f(A, B)$$

From these two equations we can clearly see that $Tf(\lambda A, \lambda B) \geq \lambda T f(A, B)$. Now, if we were to rescale the reciprocal way (i.e. with $\frac{1}{\lambda}$), we get:

$$Tf(\frac{A}{\lambda}, \frac{B}{\lambda}) \geq \frac{1}{\lambda} Tf(A, B)$$

Now, rename $a = \frac{A}{\lambda}$ and $b = \frac{B}{\lambda}$ to get:

$$Tf(a, b) \geq \frac{1}{\lambda} Tf(\lambda a, \lambda b) \rightarrow \lambda T f(a, b) \geq Tf(\lambda a, \lambda b)$$

Which, combined with the inequality we found previously implies:

$$Tf(\lambda A, \lambda B) = \lambda T f(A, B)$$

Thus, we find that $T : HOD1 \rightarrow HOD1$, and thus v is HOD1.

(b) Show that w is strictly increasing and weakly convex. With $w(1) = \frac{1}{1-\beta}$. What properties does this imply for v?

To show that w is strictly increasing we are essentially going to show that if we increase p at any rate we like, whatever increase we have will be outperformed by the operator. So consider the following: We will choose p such that $\frac{1-p}{1-a}$ is constant. This, due to the problem, implies $\frac{p-a}{1-a}$ also is constant. For simplicity let's say $\frac{1-p}{1-a} = \lambda$, so that $p = (1 - \lambda) + \lambda a$ for any $\lambda > 0$. Note here that as p increases a increases. Now, we know that as p increases it must be the case that $W(p)$ increases and $\frac{1-p}{1-a}$ remains constant. Turning to the other term, we need to ensure that $W(\frac{a}{p})$ is increasing, thus we need it to be the case that $\frac{a}{p}$ is increasing. Luckily we have an extra degree of freedom, so we will set λ such that $\frac{\partial \frac{a}{p}}{\partial a} > 0 \implies \lambda < 1$. With this choice we know that we can choose p such that the RHS of $W(a)$ is strictly increasing, and that the max operator will always outperform us, this we know that Tf is increasing. Now we can apply Corollary 1 again to find that w is strictly increasing.

Showing convexity is much easier. Note that because W is an HOD1 transformation of V, it will suffice to show that V is convex. Looking back at T, we can see that, assuming the input functions are convex, that the operator gives us the max of a linear combination of convex functions. What's more we know that linear combinations of convex functions are convex, and that the max operator preserves convexity. Thus by Corollary 1, we know that V, and consequently W, is weakly convex.

Finally, set $p = a$, this gives us:

$$W(a) = a + \beta W(a)$$

Now plugging in $W(1)$ and solving gives us:

$$W(1) = \frac{1}{1 - \beta}$$

(c) Use the functional equation for w to verify that this conjecture is correct and to show that $\hat{a} = \frac{1}{2-\beta}$.

Plugging our conjecture in gives us:

$$W(a) = \max_{a \leq p \leq 1} \left\{ \frac{1-p}{1-a} p + \beta \left[\frac{p-a}{1-a} p \frac{\frac{a}{p}}{1-\beta} + \frac{1-p}{1-a} \frac{p}{1-\beta} \right] \right\}$$

Taking a derivative w.r.t. p:

$$\frac{1-2p}{1-a} + \beta \frac{a}{(1-\beta)(1-a)} + \beta \frac{1-2p}{(1-\beta)(1-a)} = 0$$

And solving for p gives us:

$$p* = \frac{1+\beta a}{2}$$

Now, all we have to do is check if it is indeed optimal to set $p = a$, i.e. if there exists a solution to the equation:

$$a = \frac{1+\beta a}{2}$$

Which can be trivially shown to be:

$$\hat{a} = \frac{1}{2-\beta}$$