Econ 204C
Javier Birchenall
Due date: *Tuesday, April 22th, 2019*

Problem set 2: Value Function Practice

Problem 3. *Optimal growth with linear utility.* Consider, again, the optimal growth model, but this time with linear utility. That is, the social planner maximizes

$$\sum_{t=0}^{\infty} \beta^t c_t \qquad \text{s.t.} \qquad k_{t+1} = k_t^{\alpha} - c_t, \qquad c_t,\ k_t \geq 0, \qquad k_0 > 0 \ \text{ given.}$$

(a) Prove that there exists a finite solution to this problem. To do this, show that the maximal value of utility is finite. [Hint: Start by finding the maximum level of capital in the economy from the production technology.]

Note that, with the production function given above, the maximum level of capital that could ever exist is given by

$$k_{\max} = k_{\max}^{\alpha} \qquad \Longrightarrow \qquad k_{\max} = 1.$$

Thus, consumption can be no larger than 1. It then follows that

$$\sum_{t=0}^{\infty} \beta^t c_t \leq \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}.$$

That is, utility is finite.

(b) Describe the solution to the planner's problem. In so doing, you should note the mechanical problems of applying FOCs. Further, compare the results with those of the previous two problems.

Suppose you were deciding to consume or save a particular unit of output. If you consume it, you receive 1 unit of utility (recall, utility is linear). On the other hand, if you save it, you will get $\beta f'(k_{t+1})$ utils. Thus, if $\beta f'(k_{t+1}) > 1$, you will save everything (and consume nothing). Thus, we'll have $k_{t+1} = k_t^{\alpha}$. Plugging this back into the inequality, you will save everything if $\beta f'(k_t^{\alpha}) > 1$. Because we know that this will bind for the steady state, we could write this as $f(k_t) < k^*$. From the other perspective, consider the situation where $\beta f'(k_{t+1}) < 1$. In this case, one would consume everything, and $k_{t+1} = 0$. But because $\beta f'(0) = \infty > 1$, this can't happen. If $\beta f'(k_{t+1}) = 1$, then the economy is at the steady state, and an agent is indifferent between saving and consuming.

Thus, the economy will consume nothing until the steady state is reached, and then stay at the steady state forever once reached.

(c) Note that linear utility can be thought of a special case of power utility (i.e. when $\gamma = 0$). In light of your results from previous problems, discuss any insights that happen as γ varies.

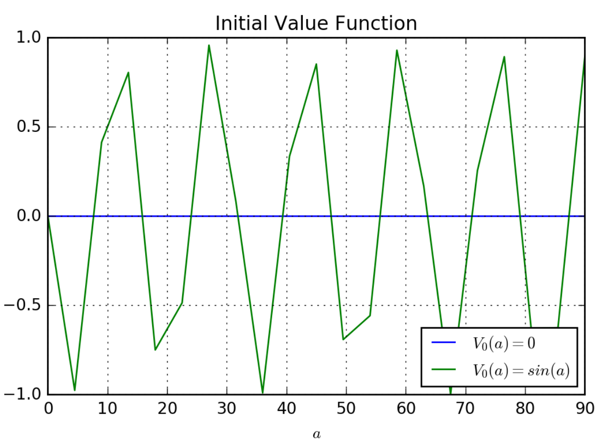
The whole story is one about savings. First, if γ is low (i.e. 0, the linear case) then agents have a high tolerance for changes in utility. Thus, the economy will converge to the steady state quickly insofar as agents will consume nothing (save everything) until it is reached. As γ increases, this tolerance disappears, with a special case when $\gamma = 1$ which coincides with log utility. Here, the income and substitution effects on savings exactly offset / cancel.

Problem 4. *Convergence of the individual's value function.* In this problem, you will write code that does value function iteration. Consider a model economy with agents who solve

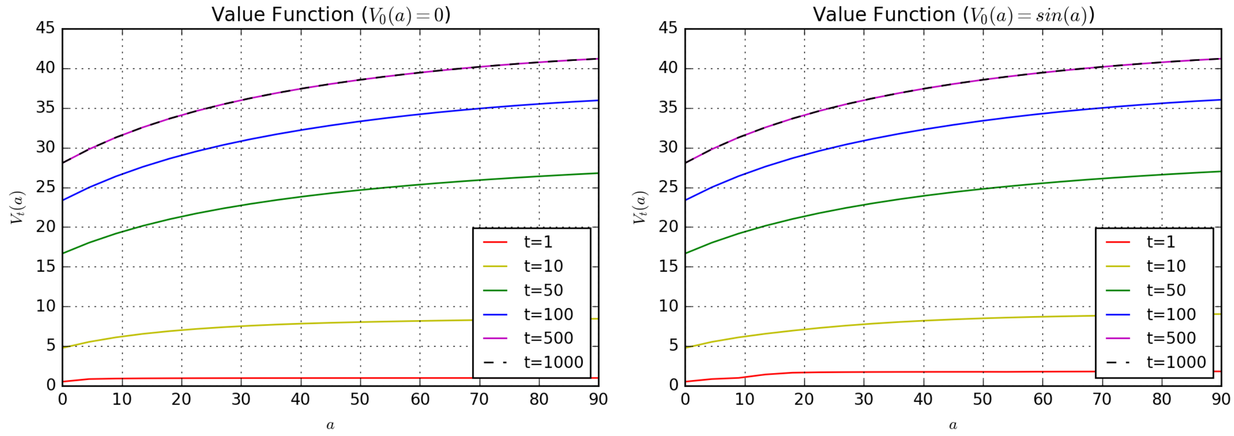
$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma} \qquad \text{s.t.} \qquad c_t + a_{t+1} = (1+r)a_t + w, \qquad c_t,\ a_t \geq 0.$$

Further, assume that we have already estimated / calibrated the parameters of the model: $\beta = 0.98$, $\gamma = 2$, $r = 0.04$, and $w = 2.09$.

(a) To start the iterative process, we need some initial $V_0(a)$. Let $V_0(a) = 0$ and, separately, let $V_0(a) = \sin(a)$. Graph both of these initial functions over a grid of possible values of a.



(b) Now write code that performs value function iteration and graph the value function at iterations 1, 10, 50, 100, 500, and 1000 (all on the same graph). Do this twice, once for both initializing functions. [Hint: To get started, you might consider writing out the Bellman equation for the problem above to better understand what's going on in the code you have been provided.]


Value Function ($V_0(a) = 0$)
$V_t(a)$
a
t=1
t=10
t=50
t=100
t=500
t=1000
Value Function ($V_0(a) = sin(a)$)
$V_t(a)$
a
t=1
t=10
t=50
t=100
t=500
t=1000