Econ 204C
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Problem set 3: RBC Models

Problem 1. Consider a large measure of infinitely-lived households solve the following problem.

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \qquad \text{s.t.} \qquad c_t + k_{t+1} = z_t n_t^{\alpha} k_t^{1-\alpha}$$

z_t is the technology shock that is assumed to have the Markov property. On top of the Cobb-Douglas production function, the utility function is assumed to be

$$u(c_t, n_t) = \theta \ln(c_t) + (1-\theta) \ln(1-n_t),$$

where c_t is consumption and n_t is hours worked.

(a) Solve for the optimal choices of c_t, n_t, and k_{t+1}. Hint: Conjecture that n_t is constant.

We can write the this problem recursively.

$$V(k,z) = \max_{c,n,k'} \left\{ \theta \ln(c) + (1-\theta)\ln(1-n) + \beta \mathbb{E}\left[V(k',z')|z\right] \right\}$$

$$\text{s.t.}$$

$$c + k' = z n^{\alpha} k^{1-\alpha}$$

The Lagrangian is

$$\mathcal{L} = \theta \ln(c) + (1-\theta)\ln(1-n) + \beta \mathbb{E}\left[V(k',z')|z\right] + \lambda\left[z n^{\alpha} k^{1-\alpha} - c - k'\right].$$

The FOCs are thus

$$\frac{\partial \mathcal{L}}{\partial c} = 0 \qquad \Longrightarrow \qquad \lambda = \frac{\theta}{c}$$

$$\frac{\partial \mathcal{L}}{\partial n} = 0 \qquad \Longrightarrow \qquad \frac{1-\theta}{1-n} = \lambda \alpha z n^{\alpha-1} k^{1-\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial k'} = 0 \qquad \Longrightarrow \qquad \lambda = \beta \mathbb{E}\left[\frac{\partial V(k',z')}{\partial k'}\Big| z\right].$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \qquad \Longrightarrow \qquad c + k' = z n^{\alpha} k^{1-\alpha}$$

We will need to use the envelope theorem.

$$\frac{\partial \mathcal{L}}{\partial k} = \frac{\partial V(k,z)}{\partial k} \qquad \Longrightarrow \qquad \frac{\partial V(k,z)}{\partial k} = \lambda(1-\alpha)zn^{\alpha}k^{-\alpha}$$
$$\Longrightarrow \qquad \frac{\partial V(k',z')}{\partial k'} = \lambda'(1-\alpha)z'n'^{\alpha}k'^{-\alpha}$$

Plug this result into the third FOC.

$$\lambda = \beta \mathbb{E}\left[\lambda'(1-\alpha)z'n'^{\alpha}k'^{-\alpha}\right]$$

Now let's combine the first FOC with the second to give an intratemporal optimality condition (for an interior).

$$\frac{1-\theta}{1-n} = \frac{\theta}{c}\alpha z n^{\alpha-1}k^{1-\alpha}$$

Combining the first and third yield the Euler equation (note we had to push forward λ to plug in as well).

$$\frac{1}{c} = \beta \mathbb{E}\left[\frac{1}{c'}(1-\alpha)z'n'^{\alpha}k'^{-\alpha}\right]$$

This is where we would normally continue on with some sort of steady state analysis. Note, however, that for this particular setup, there is actually a closed form for the choice variables.

It turns out, with Cobb-Douglas production and this particular utility function (in addition to full depreciation), the income and substitution effects of a wage rate change cancel. That is, the choice of leisure won't change.

Thus, we are going to guess (and then verify) that $n = \overline{n}$ is a constant in the solution. This guess has the implication that both c and k' are proportional to output. That is,

$$c = x_1 z k^{1-\alpha}$$
$$k' = x_2 z k^{1-\alpha},$$

where the $\overline{n}$ is absorbed by the proportion parameters x_1 and x_2. Now, plug our guesses for c and c' (obtained by pushing forward) into the Euler Equation (and then simplify).

$$\frac{1}{x_1 z k^{1-\alpha}} = \beta \mathbb{E}\left[\frac{(1-\alpha) z' n'^{\alpha} k'^{-\alpha}}{x_1 z' k'^{1-\alpha}}\right]$$

$$\frac{1}{x_1 z k^{1-\alpha}} = \beta \mathbb{E}\left[\frac{(1-\alpha) n'^{\alpha}}{x_1 k'}\right]$$

Now plug in the guess for k' (and then simplify).

$$\frac{1}{x_1 z k^{1-\alpha}} = \beta \mathbb{E}\left[\frac{(1-\alpha) n'^{\alpha}}{x_1 x_2 z k^{1-\alpha}}\right]$$

$$1 = \beta \mathbb{E}\left[\frac{(1-\alpha) n'^{\alpha}}{x_2}\right]$$

Recall that we assumed n is constant. That is, we have n' is also constant. Plugging in $n' = \overline{n}$ into the last statement (and noting that the expectation operator disappears), we can solve for x_2.

$$x_2 = (1-\alpha)\beta \overline{n}^{\alpha}$$

Next, plug the guesses for c and k' into the budget constraint.

$$x_1 z k^{1-\alpha} + x_2 z k^{1-\alpha} = z \overline{n}^{\alpha} k^{1-\alpha}$$

$$x_1 + x_2 = \overline{n}^{\alpha}$$

We can then easily solve for x_1.

$$x_1 = [1 - (1-\alpha)\beta] \overline{n}^{\alpha}$$

Now, with x_1 and x_2, we can derive explicit equations for c and k' by plugging them back into the guess.

$$c = [1 - (1-\alpha)\beta] z \overline{n}^{\alpha} k^{1-\alpha}$$

$$k' = (1-\alpha)\beta z \overline{n}^{\alpha} k^{1-\alpha}$$

Alas, we are not quite done. We have forgotten to *verify* that our initial guess what correct. To do this, plug the answer for c into the intratemporal optimality condition (the only condition we have yet to use).

$$\frac{1-\theta}{1-\overline{n}} = \frac{\alpha\theta z\overline{n}^{\alpha-1}k^{1-\alpha}}{[1-(1-\alpha)\beta]z\overline{n}^{\alpha}k^{1-\alpha}}$$

Noting all of the cancellation, we can solve the above for $\overline{n}$.

$$\overline{n} = \frac{\alpha\theta}{\alpha\theta + (1-\theta)[1-\beta(1-\alpha)]}$$

And so we have verified that our guess was correct!

Problem 2. *Simulating an economy.* Consider the simple RBC economy from the previous question. In addition, suppose that the shocks follow an AR(1) process

$$\ln(z') = \rho\ln(z) + \varepsilon$$

where $\varepsilon \sim N(\mu, \sigma^2)$. Suppose also that you've obtained the following estimates: $\alpha = .6$, $\beta = .97$, $\theta = .8$, $\mu = 0$, $\rho = .95$, and $\sigma = .4$.

(a) Generate a sequence of shocks for z for 268 periods and plot c, k, and z. Calculate the correlation of consumption and investment (k') of your simulation. Assume that the economy starts at the steady state.

See Code.

(b) Download the corresponding series from FRED and de-trend using the HP filter with $\lambda = 1600$ (for quarterly data). Calculate the correlation and compare with the model.

See Code.