Econ 204C
Javier Birchenall
Due date: *Friday, May 17th, 2019*

Problem set 4: Arrow-Debreu Markets

Problem 1. Suppose we have a market with an Arrow-Debreu structure with complete markets in dated contingent claims all traded at time 0. In each period $t \geq 0$ there is a realization of some stochastic event $s_t \in S$. Denote the history of events up to time t as $s^t = [s_0, s_1, ...s_t]$. The unconditional probability of observing a particular series of events is given by $\pi_t(s^t)$.

There are I agents named $i = 1, ..., I$, each of whom owns a stochastic endowment of one good $y_t^i(s^t)$. The history s^t is publicly observable. Household i purchases a history-dependent consumption plan $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ and orders these consumption streams by:

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)]\pi_t(s^t)$$

where $0 < \beta < 1$ and $u(c)$ is an increasing, twice continuously differentiable, strictly concave function of consumption $c \geq 0$ and satisfies the Inada Conditions. Finally, note that a feasible allocation satisfies:

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t)$$

Note: In this structure, at time $t = 0$ households can exchange claims on time t consumption, contingent on history s^t at price $q_t^0(s^t)$. For each of the following find the optimal $c_t^i(s^t)$:

(a) First, suppose that:

$$u(c) = (1-\gamma)^{-1}c^{1-\gamma}, \qquad \gamma > 0$$

Note the following is the household's budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)c_i^t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)y_t^i(s^t) \tag{1}$$

Using (1) we can set up a lagrangian for the household's problem to get the following FOC:

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) = \mu_i q_t^0(s^t) \tag{2}$$

where μ is the Lagrange multiplier to each household's budget constraint. Now, note that (2) implies:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\mu_i}{\mu_j} \tag{3}$$

i.e. that the ratio of marginal utilities between pairs of agents must be constant across all histories and times. Plugging in our utility function (u) to (3) gives us:

$$(c_t^i(s^t))^{-\gamma} = (c_t^j(s^t))^{-\gamma}\frac{\mu_i}{\mu_j}$$

Which we can rewrite as:

$$c_t^i(s^t) = c_t^j(s^t)\left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}} \tag{4}$$

Thus we can see that consumption allocations to individuals are constant fractions of one another. This means that individual allocations are perfectly correlated with the aggregate endowment, and that the fractions of the aggregate endowment assigned to each individual are independent of (s^t). One can go on to solve for the fraction α_i of the aggregate endowment each individual receives, finding that:

$$\alpha_i = \frac{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t)}{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\bar{y}_t(s^t)}$$

thus each individual's fixed consumption share is equal to its share of aggregate wealth evaluated on the competitive equilibrium pricing vector. For more details see Ljungqvist and Sargent.

(b) Now, for a general u(c) suppose that the stochastic event s^t takes values on the interval $[0, 1]$. There are two households with $y_t^1(s^t) = s_t$ and $y_t^2(s^t) = 1 - s_t$.
Note first that aggregate endowment is constant and equal to 1. We thus know that consumption will be constant: $c_t^i(s^t) = \bar{c}^i$ for all t. Now, equation (2) gives us:

$$q_t^0(s^t) = \beta^t \pi_t(s^t)\frac{u'(\bar{c}^i)}{\mu_i} \tag{5}$$

Thus the budget constraint implies:

$$\frac{u'(\bar{c}^i)}{\mu_i}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)\left[\bar{c}^i - y_t^i(s^t)\right] = 0$$

Which, upon solving for $\bar{c}^i$, gives us:

$$\bar{c}^i = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)y_t^i(s^t)$$

(c) Consider the same economy as part (b), but with the following caveat: s_t is now deterministic, alternating between 0 and 1. Thus $s_0 = 1, s_t = 0$ for t odd and $s_t = 1$ for t even.
Note in this case the probability of each agent's history occurring is 1. Thus the price system is:

$$q_t^0(s^t) = \beta^t$$

using the time 0 good as the numeraire. Thus equation (2) gives us:

$$\bar{c}^1 = (1-\beta)\sum_{j=0}^{\infty}\beta^{2j} = \frac{1}{1+\beta}$$

$$\bar{c}^2 = (1-\beta)\beta\sum_{j=0}^{\infty}\beta^{2j} = \frac{\beta}{1+\beta}$$

Note that consumer 1 consumes more in every period due to receiving the first endowment earlier.