Econ 204C
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Problem set 5

Income Fluctuations with CARA Utility

Problem 1. Solving for Consumption. You're asked to study an optimal savings plan when households face fluctuating income. The exponential (or CARA) utility function is tractable and it allows for closed-form solutions using a guess-and-verify method. Consider an agent with the following utility maximization problem:

$$\mathbb{E}\sum_{t=1}^{\infty}\left(\frac{1}{1+\delta}\right)^t u(c_t) \tag{1}$$

subject to

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0,\sigma) \tag{2}$$

$$\delta > 0, \qquad 0 < \phi < 1, \tag{3}$$

where utility takes the CARA form $u(c) = -\frac{1}{\theta}e^{-\theta c}$.

(a) The recursive formulation of this problem is given by

$$V(A,y) = \max_c \{u(c) + \beta\mathbb{E}[V(A',y')]\} \tag{4}$$

$$\text{s.t.} \qquad A' = (1+r)A + y - c. \tag{5}$$

Take the first-order condition in consumption and solve for the within period relationship between assets and consumption.

We can construct:

$$\mathcal{L}(A,y) = -\frac{1}{\theta}e^{-\theta c} + \beta\mathbb{E}[V(A',y')] + \lambda[(1+r)A - A' + y - c].$$

And then taking the F.O.C. w.r.t. consumption:

$$\frac{\partial\mathcal{L}}{\partial c} = e^{-\theta c} - \lambda = 0 \Longrightarrow \lambda = e^{-\theta c}.$$

We also know the marginal value obtained from current assets:

$$\frac{\partial\mathcal{L}}{\partial A} = \lambda(1+r).$$

Thus combining the two we obtain a relationship between assets and consumption within a period:

$$\boxed{\frac{\partial \mathcal{L}}{\partial A} = (1+r)e^{-\theta c}.}$$

(b) Guess that the value function takes the form

$$V(A, y) = -\frac{1}{\theta r}e^{-\theta r(A+ay+\bar{b})}. \tag{6}$$

Using the relationship you derived in part (a), show that the candidate optimal consumption rule takes the form

$$c^* = r(A + ay + a_0), \tag{7}$$

where we define

$$a_0 = \bar{b} + \frac{1}{\theta r}ln(1+r). \tag{8}$$

Note that $a = \frac{1}{1+r-\phi_1}$, which means that ay is the present value of human wealth given by

$$h_t = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t = \frac{y_t}{1+r-\phi_1}. \tag{9}$$

First, we can invoke the envelope theorem to say that $\frac{\partial \mathcal{L}}{\partial A} = \frac{\partial V}{\partial A}$. That is, using our guess in (6), we find that

$$\frac{\partial \mathcal{L}}{\partial A} = e^{-\theta r(A+ay+\bar{b})}.$$

We can then plug this into our answer in part (a) and do some algebra...

$$\begin{aligned}
e^{-\theta r(A+ay+\bar{b})} &= (1+r)e^{-\theta c} \\
-\theta r(A+ay+\bar{b}) &= -\theta c + ln(1+r) && \text{(taking log of both sides)} \\
c^* &= r(A+ay+\bar{b}) + \frac{1}{\theta}ln(1+r) && \text{(solving for } c^*\text{)} \\
&= r(A+ay+\bar{b}+\frac{1}{\theta r}ln(1+r)) \\
&= \boxed{r(A+ay+a_0)} && \text{(plugging in for } a_0\text{)}
\end{aligned}$$

(c) Using our guess of the value function, we can rewrite the Bellman Equation as

$$V(A, y) = \frac{r}{1+r}V(A, y) - \left(\frac{1}{1+\delta}\right)\frac{1}{\theta r}\mathbb{E}\left[exp\left(-\theta r\left(A' + ay' + \bar{b}\right)\right)\right]. \tag{10}$$

Plug in the equation for the evolution of assets for A' and the AR(1) process that determines income for y', as well as your guess for V, and show that consumption is equal to

$$c = r\left\{A + \frac{1-a+a\phi_1}{r}y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln\left(\mathbb{E}\left[exp\left(-\theta r a\varepsilon'\right)\right]\right)\right]\right\}. \quad (11)$$

(Two hints: 1. Derivatives are not required!; 2. Remember that $exp(a+b) = exp(a)\times exp(b)$)

We start by subtracting the $V(A,y)$ term on the right over to get it to simplify *slightly.*

$$\left(\frac{1}{1+r}\right)\left(\frac{-1}{\theta r}\right)exp\{-\theta r(A+ay+\bar{b})\}$$
$$= \left(\frac{1}{1+\delta}\right)\left(\frac{-1}{\theta r}\right)\mathbb{E}\left[exp\left\{-\theta r\left[(1+r)A+y-c+a(\phi_0+\phi_1 y+\varepsilon')+\bar{b}\right]\right\}\right]$$

Next, note that the expectation term simplifies. We can pull out things that are already determined (i.e. things without primes on them):

$$\implies exp\left\{-\theta r\left[(1+r)A+y-c+a\phi_0+a\phi_1 y+\bar{b}\right]\right\}\mathbb{E}\left[exp\left\{-\theta r a\varepsilon'\right\}\right].$$

Now we can take the ln of both sides:

$$-\theta r(A+ay+\bar{b}) - \ln\left(\frac{1+\delta}{1+r}\right) = -\theta r[(1+r)A+y-c+a\phi_0+a\phi_1 y+\bar{b}] + \ln(\mathbb{E}[exp\{-\theta r a\varepsilon'\})].$$

Isolating the c term:

$$\theta r c = \theta r[(1+r)A+y+a\phi_0+a\phi_1 y+\bar{b}] - \theta r(A+ay+\bar{b}) + \ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp\{-\theta r a\varepsilon'\}]).$$

Dividing by θr and simplifying:

$$c = rA + (1-a+a\phi_1)y + a\phi_0 + \frac{1}{\theta r}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln\left(\mathbb{E}\left[exp\left\{-\theta r a\varepsilon'\right\}\right]\right)\right].$$

$$\implies \boxed{c = r\left\{A + \frac{1-a+a\phi_1}{r}y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln\left(\mathbb{E}\left[exp\left(-\theta r a\varepsilon'\right)\right]\right)\right]\right\}}.$$

(d) Using the method of undetermined coefficients (i.e. set your two solutions for consumption equal), solve for $\bar{b}$ using your solution obtained in part (b).

First we take our first solution for c given by (7) and plug in for a_0 given by (8). Then, setting this equal to our last solution for consumption:

$$r(A + ay + \bar{b} + \frac{1}{\theta r}\ln(1+r))$$
$$= r\{A + \frac{1-a+a\phi_1}{r}y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a\varepsilon')])]\}$$

Grouping terms together:

$$\bar{b} = \left(\frac{1-a+a\phi_1}{r} - a\right) y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - r\ln(1+r)\right]$$

$$= \underbrace{\left(\frac{1-a\overbrace{(1+r-\phi_1)}^{=1/a}}{r}\right)}_{=0} y + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - r\ln(1+r)\right]$$

$$\implies \boxed{\bar{b} = \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - rln(1+r)\right]}$$

(e) Show that this solution for consumption can be written as

$$c^* = r(A + h - \Gamma(r)), \tag{12}$$

where $h = a(y + \frac{\phi_0}{r})$ is human wealth and $\Gamma(r) = \frac{1}{\theta r^2}[ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - ln(\frac{1+\delta}{1+r})]$ is the difference between precautionary savings and impatience caused by a distaste for lower consumption.

Recall that our expression for consumption was given by $c = r(A + ay + \bar{b} + \frac{1}{\theta r} ln(1+r))$ (plug (8) into (7)). Plugging what we found for $\bar{b}$ and simplifying:

$$c^* = r\{A + ay + \frac{a\phi_0}{r} + \frac{1}{\theta r^2}\left[\ln\left(\frac{1+\delta}{1+r}\right) - \ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - \cancel{r\ln(1+r)}\right] + \cancel{\frac{1}{\theta r}\ln(1+r)}\}$$

$$= r\left\{A + \underbrace{a\left(y + \frac{\phi_0}{r}\right)}_{h} - \underbrace{\frac{1}{\theta r^2}\left[\ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - \ln\left(\frac{1+\delta}{1+r}\right)\right]}_{\Gamma(r)}\right\}$$

$$\implies \boxed{c^* = r(A + h - \Gamma(r))}$$

Problem 2. General Equilibrium. We can write the savings rate as

$$s^* = rA^* + y - c^* \tag{13}$$

(a) Show that this can be rewritten as

$$s^* = f + \frac{1}{\theta r}\left[\ln(\mathbb{E}[exp(-\theta a r \varepsilon')]) - \ln\left(\frac{1+\delta}{1+r}\right)\right]. \tag{14}$$

(Hint: plug in the expression that you just found for consumption and then let f be the other terms.)

Plugging in what we found for consumption:

$$\begin{aligned} s^* &= rA^* + y - r(A + h - \Gamma(r)) \\ &= \underbrace{r(A^* - A) + y - rh}_{f} + r\Gamma(r) \\ &= \boxed{f + \frac{1}{\theta r}\left[\ln(\mathbb{E}[exp(-\theta a r \varepsilon')]) - \ln\left(\frac{1+\delta}{1+r}\right)\right].} \end{aligned}$$

(b) At the aggregate level, "rainy-day savings" are 0 ($\mathbb{E}(f) = 0$). Thus, aggregate savings becomes

$$S(r) = \frac{1}{\theta r}\left[\ln(\mathbb{E}[exp(-\theta a r \varepsilon')]) - \ln\left(\frac{1+\delta}{1+r}\right)\right]. \tag{15}$$

Show that $r = \delta$ cannot be an equilibrium with $S(r) = 0$. Prove that for an equilibrium, $r < \delta$ is a necessary condition.

We will use Jensen's inequality in this problem. For review, it is given below.

Jensen's Inequality: For a concave function g, $g(\mathbb{E}(X)) \leq \mathbb{E}(g(X))$. Where the inequality is strict if the function is strictly concave.

Taking the expression in (14) and plugging in δ for r:

$$\begin{aligned} S(\delta) &= \frac{1}{\theta \delta}\left[\ln(\mathbb{E}[exp(-\theta a \delta \varepsilon')]) - \cancelto{0}{ln\left(\frac{1+\delta}{1+\delta}\right)}\right] \\ &= \frac{1}{\theta \delta}\ln(\mathbb{E}[exp(-\theta a \delta \varepsilon')]). \end{aligned}$$

Now, we can use Jensen's inequality on the above result:

$$\begin{aligned} S(\delta) &= \frac{1}{\theta \delta}\ln(\mathbb{E}[exp(-\theta a \delta \varepsilon')]) \\ &> \frac{1}{\theta \delta}\mathbb{E}(\ln[exp(-\theta a \delta \varepsilon')]) \\ &= \frac{1}{\theta \delta}\mathbb{E}(-\theta a \delta \varepsilon') \\ &= -a\mathbb{E}(\varepsilon') = 0. \end{aligned}$$

Thus we have shown that $S(r) > 0$ if $r = \delta$. And so this can't be an equilibrium with 0 aggregate savings. Intuitively, if the interest rate covers exactly the rate of depreciation, individuals will need to save positive amounts in the aggregate in order to be able to insure themselves with asset dividends in the presence of uncertainty.

In order for aggregate savings to be 0 (that is, in order for the total amount of assets to equal the total amount of debt) so that individuals don't save too much or too little in the aggregate, how should r and δ be related? To find out, we'll do similar calculations as before, but this time keeping r general:

$$\begin{aligned} S(r) &= \frac{1}{\theta r}\left[\ln(\mathbb{E}[exp(-\theta a r \varepsilon')]) - \ln\left(\frac{1+\delta}{1+r}\right)\right] \\ &> \frac{1}{\theta r}\left[\mathbb{E}(-\theta a r \varepsilon') - \ln\left(\frac{1+\delta}{1+r}\right)\right] \\ &= -\frac{1}{\theta r}\ln\left(\frac{1+\delta}{1+r}\right). \qquad (\star) \end{aligned}$$

Now suppose that $r > \delta$. The $\ln(\cdot)$ argument in $(\star)$ will be negative, so the whole term will be positive. That is, $S(r) > 0$. Thus, in order for aggregate savings to be 0, the interest rate r cannot exceed the rate of depreciation δ. Intuitively, if the interest rate earned on assets is sufficiently high, agents will save not only to insure against bad shocks in the future, but also because the returns are so high that it's in their best interest to "over-save."

Remember that this is all in the context of a steady-state (or equilibrium), so we are looking for conditions that would result in roughly constant levels of effective unit quantities. What we have here is too much savings that would lead to a growth in the total level of assets in the economy over time, and thus not in equilibrium.

Now suppose that $r < \delta$. Then $ln(\cdot)$ will be positive, so $(\star)$ will be negative. Denote whatever value this takes as $\underline{S}$. Thus we will have that $S(r) > \underline{S}$, where $0 \in (\underline{S}, 1)$. And so it must be that $r < \delta$ in order for it to be possible for the economy to be in equilibrium (which necessitates $S(r) = 0$ *a la* discussion above).

(c) Using this knowledge of the aggregate savings rate, argue that agents in this economy exhibit consumption that satisfies the permanent income hypothesis. In other words, show that consumption is composed of two terms: the asset dividend and the dividend from the present value of human wealth. That is,

$$c^* = r(A + h) \qquad (16)$$

We have just argued that, in equilibrium, we must have $S(r) = 0$. We have also shown that in order for this to be possible, we need $r < \delta$. Also recall that we found that consumption was given by

$$
\begin{aligned}
c^* &= r(A + h - \Gamma(r)) \\
&= r\Big\{A + h - \left(\frac{1}{r}\right) \underbrace{\frac{1}{\theta r}\Big[\ln(\mathbb{E}[exp(-\theta r a \varepsilon')]) - \ln\left(\frac{1+\delta}{1+r}\right)\Big]}_{S(r)=0}\Big\} \\
&= \boxed{r(A+h)}
\end{aligned}
$$

In words, if the economy is in equilibrium, aggregate savings will be 0. In this context, this is also equivalent to saying that the average savings will be 0. And so (on average), an agent's consumption will be composed of both asset dividends and the dividend from the present value of human wealth, and so will satisfy the permanent income hypothesis.