Econ 204C: Section 7

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Where We're Headed

- In last week's section we introduced a basic savings problem with idiosyncratic shocks, focusing on the household's side
- Now we will think about what an agent's savings are doing and what the appropriate equilibrium definition is
- You should note that while we sometimes refer to these equilibrium concepts by the same name (e.g. a *stationary recursive competitive equilibrium*), the exact definition is model specific (it depends on the explicit environment we are studying)
- It is important to be able to construct the appropriate equilibrium definition: it can help organize your thoughts and it provides a crude outline of how you would go about solving the model

What We've Seen So Far

Let there be a unit measure of households that maximize the present discounted value of lifetime utility over consumption.

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + a_{t+1} = (1 + r)a_t + ws_t \quad \forall t$$
$$a_t \geq -\phi \quad \forall t$$

$\beta \in (0, 1)$ is the discount factor, $\phi > 0$ is some borrowing constraint, and $u(c_t)$ is an increasing, strictly concave, twice continuously differentiable utility function that satisfies the Inada conditions.

s_t is some idiosyncratic finite state Markov shock with transition probabilities denoted by $\mathcal{P}(i, j) \equiv Prob(s' = s_j | s = s_i)$.

Suppose that we have realized a state s_i today. The value function for a household can be written as

$$V(a, s_i) = \max_{a'} \left\{ u\Big((1+r)a + ws_i - a'\Big) + \beta \sum_{j=1}^{m} V(a', s_j)\mathcal{P}(i,j) \right\}$$

$$\text{s.t.} \quad a' \geq -\phi.$$

The policy function $a' = g(a, s)$ and the transition probabilities will induce a law of motion for the distribution of (a, s) across the economy over time.

$$\lambda_{t+1}(a', s') = \sum_{s} \int \mathbb{I}\big(g(a,s) \in \mathcal{A}\big)\mathcal{P}(s, s')d\lambda_t(a, s)$$

Notice that, in contrast to last time, we are allowing the support of a to be continuous.

- If our Markov chain has the "correct" properties (for finite state Markov process, it should be irreducible), there will exist an invariant distribution $\lambda^*(a, s)$ that doesn't change over time

- There are more general conditions that will guarantee that an invariant distribution exists, should they be needed
  - a continuous state Markov requires irreducibility and positive recurrence
  - more generally, if a stochastic process is *monotone*, satisfies the *Feller property*, and satisfies the *mixing condition*, then there will exist a unique invariant distribution

- To be clear, individuals will move around in the stationary distribution because of shocks, but the distribution itself remains the same

Another interesting object of study in these models is the population mean assets. Given some interest rate r (and wage w), the population mean is given by

$$\mathbb{E}_r[a'] = \int g(a,s) d\lambda^*(a,s).$$

Because the *policy* function is the same for everyone, we can use it to index the population over the state vector (a, s). Integrating over $g(\cdot)$ with the distribution λ^* will thus give the mean assets in the economy tomorrow. Further, because λ^* an invariant distribution, the mean today is the same as the mean tomorrow:

$$\mathbb{E}_r[a] = \mathbb{E}_r[a'].$$

Now, what is r (and w), and how is it (are they) determined?

Huggett (1993)

- Each household has access to a centralized loan market where it may borrow / lend at a constant interest risk-free rate r
- Each household has an endowment that is governed by, for example, some finite state space for s and its associated transition probabilities, $\mathcal{P}$
- Since this is an endowment economy, we can set $w = 1$ and let the state space of s absorb any variability in income that we might desire
- Borrowing by the household may not exceed $\phi > 0$
  - because the interest rate is assumed to be risk-free, households must be able to pay back the loan with probability 1 (and so there will be some underlying notion of a natural debt limit)

Stationary RCE

Given a borrowing limit ϕ, a *stationary recursive competitive equilibrium* in the above environment is an interest rate r, a value function $V(a,s)$, a policy function $g(a,s)$, and a distribution $\lambda^*(a,s)$ such that

1. Given r, $V(a,s)$ and $g(a,s)$ solve the household's problem
2. The loan market clears

$$\int g(a,s)d\lambda^*(a,s) = 0$$

3. The probability distribution $\lambda^*(a,s)$ is the invariant distribution of (a,s) induced by the stochastic process for s and the optimal policy function $g(a,s)$.

Computational Algorithm

As you might have known, much of what we are doing cannot be solved analytically. How would one go about solving the above model? One of the most common ways is to iterate on the prices (here it's just r).

For $j = 0, 1, \ldots$

1. Fix $r_{(j)}$ and solve the household's problem for the policy (and value) functions $g_{(j)}(a, s)$ (and $V_{(j)}(a, s)$)
2. Determine the associated stationary distribution $\lambda^*_{(j)}(a, s)$
3. Use the above to determine what the excess demand, e, for loans is

$$e_{(j)} = \int g_{(j)}(a, s,)d\lambda^*_{(j)}(a, s)$$

4. If $e^*_{(j)} > 0$, the interest rate is too high (there is too much saving) and so we should set $r_{(j+1)} < r_{(j)}$ and repeat; the opposite if $e_{(j)} < 0$; stop if $e_{(j)} = 0$

Aiyigari (1994)

- Here, the single asset that household's may purchase is interpreted as productive *capital* that evolves according to

$$k_{t+1} = (1 - \delta)k_t + i_t$$

- The budget constraint of a household is given by

$$c_t + i_t = (1 + r)k_t + ws_t$$

- Employment (s_t) is a stochastic as before and, for example, follows a finite state Markov process with transition probabilities $\mathcal{P}$

- Under the appropriate assumptions, there will exist an invariant distribution $\lambda^*(k, s)$

- The key here is that individuals will experience different employment histories, $s_0^t = \{s_h\}_{h=0}^t$, and will therefore have accumulated different amounts of capital at any point in time

- Assume that the economy is in its invariant distribution; the total (and average) level of capital is given by

$$K = \int g(k, s) d\lambda^*(k, s)$$

- N, the total level of employment, similarly will be given by the average employment level in the economy

$$N = \int s d\lambda^*(k, s)$$

- There exists an aggregate CRS production function $F(K, N)$ that determines the prices r and w from the firm's FOCs:

$$r = \frac{\partial F(K, N)}{\partial K} - \delta \qquad w = \frac{\partial F(K, N)}{\partial N}$$

Stationary RCE

A *stationary recursive competitive equilibrium* in the above environment is a value function $V(k,s)$, a policy function $g(k,s)$, a distribution $\lambda^*(k,s)$, prices r and w, and a level of capital K such that

1. Given r and w, $V(k,s)$ and $g(k,s)$ solve the household's problem
2. Given r and w, K and N maximize profits
3. The capital and labor markets clear

$$K = \int g(k,s) d\lambda^*(k,s) \qquad\qquad N = \int s d\lambda^*(k,s)$$

4. The probability distribution $\lambda^*(k,s)$ is the invariant distribution of (k,s) induced by the stochastic process for s and the optimal policy function $g(k,s)$

Computational Algorithm

There are many ways to tackle this computationally. Note that the average level of employment is a free variable that can be normalized to something easy: $N = 1$. Here, similar to before, we can iterate on the prices (r, w).

For $j = 0, 1, \ldots$

1. Fix $r_{(j)}$ and find the associated $w_{(j)}$, then solve the household's problem for the policy (and value) functions $g_{(j)}(k, s)$ (and $V_{(j)}(k, s)$)

2. Determine the associated stationary distribution $\lambda^*_{(j)}(k, s)$

3. Use the above to calculate the aggregate level of capital in the economy (note that N won't change iteration by iteration)

$$K = \int g(k, s) d\lambda^*(k, s)$$

4. Calculate the interest rate implied by K and N, r^*, and check to see if $r = r^*$; if it's not, adjust r accordingly and continue

Krusell & Smith (1998)

- It turns out, calibrating the Aiyigari model doesn't produce *enough* wealth dispersion
  - extra savings are not important enough for the wealthy
- Krusell & Smith extend the Aiyigari model to include aggregate (in addition to idiosyncratic) uncertainty
- This uncertainty is captured by shocks to the production function; output is now given by $zF(K, N)$
- Now, note that there will not be a stationary distribution λ^* because of these aggregate shocks
- Further, because λ helps in determining prices, they'll need to be able to forecast what λ will be tomorrow

- That is, at some point in time t, agents will need to keep track of z and λ as state variables, and will also need to know how they evolve over time

- The household's problem can be expressed by

$$V(a, s, z, \lambda) = \max_{c,k'} \{u(c) + \beta \mathbb{E}[V(k', s', \lambda', z') | s, z, \lambda]\}$$
$$\text{s.t.} \quad c + k' = (1 + r)k + ws$$
$$k' \geq 0$$
$$\lambda' = H(\lambda, z)$$

- Profit maximization from the firm's problem implies that

$$r = z\frac{\partial F(K, N)}{\partial K} - \delta \qquad w = z\frac{\partial F(K, N)}{\partial N}$$

RCE

A *recursive competitive equilibrium* in the above environment is a value function $V(k, s, \lambda, z)$, a policy function $g(k, s, \lambda, z)$, pricing functions $r(\lambda, z)$ and $w(\lambda, z)$, and a law of motion for λ given by $H(\lambda, z)$ such that

1. Given $r(\lambda, z)$ and $w(\lambda, z)$, $V(k, s, \lambda, z)$ and $g(k, s, \lambda, z)$ solve the household's problem
2. Given $r(\lambda, z)$ and $w(\lambda, z)$, K and N maximize profits
3. The capital and labor markets clear
$$K = \int k d\lambda(k, s) \qquad N = \int s d\lambda(k, s)$$
4. The law of motion for the distribution λ induced by the stochastic processes for s and z and the policy function $g(k, s, \lambda, z)$ is given by $H(\lambda, z)$