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Econ 204C
9 June 2017

Final Exam
Spring 2017

This exam is closed book. Most points are given for the correct set-up of a problem and for economically insightful interpretations. You have 3 hours for a maximum score of 100 points.

[30] **Problem 1.** *Market incompleteness in a quadratic, two-period economy.* Consider a large number of ex-ante identical households that live for two periods, receive a deterministic (and equal across households) income y in period one and a stochastic income Y' in the second period. Y' is distributed according to a density function $\pi(y')$ defined by $\Pi(y') \equiv \Pr(Y' \leq y') = \int_0^{y'} \pi(s)ds$.

Consumption in period one is c_i where $i = 1, ..., I$ indexes households with I being a "large" value. Consumption in period two is $c_i'(Y')$. Since households are ex-ante identical, without loss of any generality, one can omit the household i index. Households maximize an expected utility function given by

$$U(c, c'(y')) = u(c) + \beta \int_0^{\infty} u(c'(y'))\pi(y')dy',$$

where $u(c)$ is quadratic and satisfies $u(c) = c - (\phi/2)c^2$, with $\phi > 0$ and $c < 1/\phi$ so that $u_c(c) > 0$ and $u_{cc}(c) < 0$, and with a discount rate $0 < \beta < 1$. The purpose of the following questions is to examine the household's behavior under several trading arrangements.

[10] **(a)** Assume first that there are no trading arrangements to diversify income risk. Show that under autarky, the welfare of the household satisfies

$$U^A = u(y) + \beta u(\mathbb{E}[Y']) - \beta\frac{\phi}{2}Var(Y'),$$

where $\mathbb{E}[Y'] = \int_0^{\infty} y'\pi(y')dy'$. [Hint: Recall that $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.]

Under autarky, we know that $c = y$ and $c' = Y'$.

$$\begin{aligned}
U^A &= u(y) + \beta \int\limits_0^\infty u(Y')\pi(Y')dY' \\
&= u(y) + \beta \int\limits_0^\infty \Big[Y' - \frac{\phi}{2}Y'^2\Big]\pi(Y')dY' \\
&= u(y) + \beta\mathbb{E}[Y'] - \beta\frac{\phi}{2}\mathbb{E}[Y'^2] \\
&= u(y) + \beta\mathbb{E}[Y'] - \beta\frac{\phi}{2}\Big\{Var(Y') + (\mathbb{E}[Y'])^2\Big\} \\
&= u(y) + \beta\Big\{\mathbb{E}[Y'] - \beta\frac{\phi}{2}(\mathbb{E}[Y'])^2\Big\} - \beta\frac{\phi}{2}Var(Y') \\
&= u(y) + \beta u(\mathbb{E}[Y']) - \beta\frac{\phi}{2}Var(Y')
\end{aligned}$$

[10] **(b)** Assume now that a series of Arrow-Debreu markets are available. Households can trade contingent claims before the resolution of uncertainty. The price of a contingent unit of consumption in period two, for an income realization y', is $P(y')$. Write down the household's budget constraint using the price of consumption in period one as numeraire, define a competitive Arrow-Debreu equilibrium, and solve for the equilibrium under complete markets. Show that under complete markets, the welfare of households satisfies

$$U^{CM} = U^A + \beta\frac{\phi}{2}Var(Y').$$

[Hint: You might find it useful to invoke the Law of Large Numbers, as in $\lim_{I\to\infty}\frac{1}{I}\sum_{i=1}^{I} y'_i = \mathbb{E}[Y'] \equiv \int_0^\infty y'\pi(y')dy'$, to characterize the solution.]

The budget constraint is

$$c + \int_0^\infty P(y')c'(y')dy' = y + \int_0^\infty P(y')y'dy'.$$

The first-order conditions are

$$u_c(c) = \lambda \qquad \text{and} \qquad \beta\pi(y')u_c(c'(y')) = \lambda P(y').$$

The market clearing conditions are

$$c = y \qquad \text{and} \qquad \sum\nolimits_{i=1}^{I} c_i(y') = \sum\nolimits_{i=1}^{I} y'_i,$$

since β, $\pi(y')$, λ, and $P(y')$ are invariant with respect to an agent's type, the allocation must be such that $c_i'(y') = c_j'(y') = \bar{c}'$ so that

$$\bar{c}' = \frac{1}{I}\sum\nolimits_{i=1}^{I} y_i'.$$

Under the Law of Large Numbers, $\lim_{I\to\infty}\frac{1}{I}\sum_{i=1}^{I} y_i' = \mathbb{E}[Y'] \equiv \int_0^\infty y'\pi(y')dy'$. Thus, $\bar{c}' = \mathbb{E}[Y']$. Therefore, $U^{CM} = u(y) + \beta u(\mathbb{E}[Y'])$.

[10] **(c)** Finally, consider an economy where households can borrow and save a risk-less asset but markets are incomplete due to limited borrowing. Initial assets are zero. The household's budget constraints are $c + a = y$, and $c' = (1+r)a + Y'$, where a denotes the assets saved in the first period and r is the risk-free rate of return.

i. Show that when there are no borrowing constraints and $\beta(1+r) = 1$, household consumption is a *martingale*. (A martingale is a stochastic process x that satisfies $\mathbb{E}[x'] = x$.) Further, show that household assets satisfy

$$a = \frac{y - \mathbb{E}[Y']}{2+r}.$$

Answer is the same as before.

ii. Suppose now that a borrowing constraint of the form $a \geq 0$ is imposed on these households. Let $\widehat{c}$ and $\widehat{c}'$ denote the households consumption in each period under borrowing constraints. The modified first-order condition is of the form

$$\widehat{c} = \begin{cases} \mathbb{E}[\widehat{c}'] \text{ if } a > 0 \\ y \text{ if } a = 0. \end{cases}$$

Show that consumption can be written as

$$\widehat{c} = y + \min\{-a, 0\}.$$

Using the fact that $\mathbb{E}[\widehat{c}'] = (1+r)a + \mathbb{E}[Y']$, notice that the first order condition to the constrained problem can be written as $\widehat{c} = \min\{\mathbb{E}[\widehat{c}'], y\} \leq \mathbb{E}[\widehat{c}']$. The first equality gives

$$\begin{aligned} \widehat{c} - y &= \min\{\mathbb{E}[\widehat{c}'] - y, 0\} \\ &= \min\{(1+r)a + \mathbb{E}[Y'] - y, 0\} \\ &= \min\{(1+r)a - (2+r)a, 0\} \end{aligned}$$

which is the stated expression.

iii. Suppose that log-income, $\ln Y'$, is normally distributed with mean μ and variance σ^2. The mean of income is $\mathbb{E}[Y'] = \exp\{\mu + \sigma^2/2\}$. Suppose also that $\exp\mu = \rho y$, with $\rho < 1$. Show that if σ^2 is "small," the borrowing constraint does not bind. Show also that there is a critical value for σ^2 such that as $\sigma^2 \to \sigma^2_+$, the borrowing constraint binds almost surely. Explain.

Assets satisfy

$$a = \frac{y[1 - \rho\exp\{\sigma^2/2\}]}{2 + r}$$

so when $\sigma^2 = 0$, assets are positive $a(\sigma^2 = 0) = y(1-\rho)/(2+r) > 0$ and consumption is smoothed through positive savings. (Recall that future income is lower than present income by a factor ρ.) In this case the constraint is not binding so $c = \hat{c}$. As the volatility of income increases, there is a precautionary demand for savings. Households would like to borrow in order to insure themselves against a possibly low future income. In fact, there is a level of income volatility $\sigma^2_+ = -2\ln\rho > 0$ such that $a(\sigma^2_+) = 0$. These households are indifferent between borrowing or saving. More importantly, households will be borrowing constrained for any value of volatility with $\sigma^2 > \sigma^2_+$ as $a(\sigma^2 > \sigma^2_+) < 0$. The lesson is that despite the quadratic utility function, there is a demand for precautionary savings under borrowing constraints as long as $\sigma^2 \in (0, \sigma^2_+]$.

[30] **Problem 2.** *A model with search intensity.* Consider a search economy in which wages are given by an exogenous distribution $F(w)$ with a support bounded by $\underline{w}$ and $\overline{w}$. Matches are determined by an exogenous matching rate $\lambda(e)$, where $e \geq 0$ is their search effort while unemployed. $\lambda(e)$ is assumed to be strictly increasing and concave with $\lambda'(0) \to \infty$, and individuals face strictly increasing and convex search costs $c(e)$ with $c'(0) = 0$. During unemployment, agents receive flow utility b, and during employment they consume their wage. Matches end exogenously at a rate s.

[5] **(a)** Write the flow Bellman equations for the employment and unemployment states.

$$rV(w) = w + s\Big[U - V(w)\Big]$$

$$rU = \max_e \left\{ b - c(e) + \lambda(e) \int_{\underline{w}}^{\overline{w}} \max\Big[V(w) - U, 0\Big] dF(w) \right\}$$

[10] **(b)** Letting e^* denote the optimal search intensity, derive the reservation wage. Be sure to solve as extensively as possible.

The reservation wage is defined by $rV(w_R) = rU$. First, construct $rV(w_R)$.

$$rV(w_R) = w_R + s\Big[U - V(w_R)\Big] \qquad \Longrightarrow \qquad rV(w_R) = r\frac{w_R + sU}{r+s}.$$

Since we know this equals rU, we can determine that

$$r\frac{w_R + sU}{r+s} = rU \qquad \Longrightarrow \qquad w_R = rU.$$

Next, note that we can rewrite the flow value of unemployment as follows.

$$rU = b - c(e^*) + \lambda(e^*)\int_{w_R}^{\overline{w}} \Big[V(w) - U\Big] dF(w)$$

Further, we can determine that

$$V(w) - U = \frac{w + sU}{r+s} - \frac{w_R + sU}{r+s} = \frac{w - w_R}{r+s}.$$

We can use the above to solve for the reservation wage.

$$\begin{aligned}
w_R &= b - c(e^*) + \lambda(e^*)\int_{w_R}^{\overline{w}} \Big[V(w) - U\Big] dF(w) \\
&= b - c(e^*) + \frac{\lambda(e^*)}{r+s}\int_{w_R}^{\overline{w}} \Big[w - w_R\Big] dF(w) \\
&= b - c(e^*) + \frac{\lambda(e^*)}{r+s}\int_{w_R}^{\overline{w}} [1 - F(w)] dw \qquad \text{(integration-by-parts)}
\end{aligned}$$

[15] **(c)** Solve for the optimal search intensity, e^*. In doing so, you should show that the optimal search intensity maximizes the reservation wage (which is a function of e).

Rewrite the flow value of unemployment. Note that I am explicitly writing the reservation wage as a function of e. The optimal search intensity will then pin down the "optimal" reservation wage.

$$rU = \max_e \left\{ b - c(e) + \frac{\lambda(e)}{r+s} \underbrace{\int_{w_R(e)}^{\overline{w}} [1 - F(w)]dw}_{\equiv \varphi(w_R(e))} \right\}$$

Now, differentiate w.r.t. e and set equal to 0 (let primes denote derivatives).

$$-c'(e^*) + \frac{\lambda'(e^*)}{r+s}\varphi(w_R(e^*)) + \frac{\lambda(e^*)}{r+s}\varphi'(w_R(e^*))w_R'(e^*) = 0$$

We can straightforwardly determine what $\varphi'(w_R(e))$ is and use our earlier results to figure out what $w_R'(e)$ is.

$$\varphi'(w_R(e)) = -[1 - F(w_R(e))]$$

$$w_R'(e) = -c'(e) + \frac{\lambda'(e)}{r+s}\varphi(w_R(e)) + \frac{\lambda(e)}{r+s}\varphi'(w_R(e))w_R'(e).$$

Interestingly, we can see that the last statement should be equal to 0 (it's the same as the FOC).[1] That is, the optimal choice of e will maximize the reservation wage ($w_R'(e^*) = 0$). Plugging these results in to the FOC, we are left with an expression that pins down the optimal search intensity, e^*.

$$c'(e^*) = \frac{\lambda'(e^*)}{r+s}\varphi(w_R(e^*))$$

We can quickly verify that there will be a unique e^* that solves the above given the assumptions on $\lambda(\cdot)$ and $c(\cdot)$.

[40] **Problem 3.** *Temporary unemployment compensation.* At the beginning of each period an unemployed worker draws one offer to work forever at wage w (which she may accept or reject). Wages are i.i.d. draws from the c.d.f. F, where $F(\underline{w}) = 0$ and $F(\overline{w}) = 1$. The worker seeks to maximize $\mathbb{E}\sum_{t=0}^{\infty} \beta^t y_t$, where y_t is the worker's wage or unemployment compensation, if any. The worker is entitled to unemployment compensation in the amount $\gamma > 0$ only during the *first* period that she is unemployed. After one period on unemployment compensation, the worker receives none.

[1] One might also note that because it must be that $rU = w_R(e)$, maximizing rU w.r.t. e necessitates maximizing $w_R(e)$ w.r.t. e.

[10] **(a)** Write the Bellman equations for this problem.

Let a superscript "1" denote the first period of an unemployment spell, and a superscript "+" denote any future periods in such spell.

$$
\begin{aligned}
V_E(w) &= w + \beta V_E(w) \\
V_U^1(w) &= \max\left\{ V_E(w), \gamma + \beta \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w') \right\} \\
V_U^+(w) &= \max\left\{ V_E(w), \beta \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w') \right\}
\end{aligned}
$$

[10] **(b)** Show and explain how the worker's reservation wage and her "hazard of leaving unemployment" (i.e. the probability of accepting a job offer) varies with the duration of unemployment.

As usual, the reservation wage is given by the wage such that an individual is indifferent between working and remaining unemployed. Here, depending on how many periods the worker has been unemployed, the reservation wage will be different. Denote these w_R^1 and w_R^+. Further, we can rewrite $V_E(w) = \frac{w}{1-\beta}$.

$$
\begin{aligned}
w_R^1 &= (1-\beta)\gamma + (\beta - \beta^2) \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w') \\
w_R^+ &= (\beta - \beta^2) \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w')
\end{aligned}
$$

We can see that $w_R^1 - w_R^+ = (1-\beta)\gamma > 0$. That is, the reservation wage is decreasing as an unemployment spell continues. In the first equation above, the reservation wage is proportional to unemployment benefits *plus* the option value of waiting for another offer (where one knows that there will be no benefits in the future).

Next, note that the hazard of leaving unemployment is given by the probability that a wage is drawn at least as large as the reservation wage.

$$
\Pr(w \geq w_R^i) = 1 - F(w_R^i) \qquad i \in \{1, +\}
$$

Noting the discussion above about the result that $w_R^1 > w_R^+$, we can easily determine that $\Pr(w \geq w_R^1) \leq \Pr(w \geq w_R^+)$, that is the hazard of leaving unemployment is increasing with the length of an unemployment spell.

Now assume that the worker is also entitled to unemployment compensation if she quits a job. As before, the worker receives unemployment compensation in the amount of γ during the first period of an unemployment spell, and zero during the remaining part of the spell. (In order to re-qualify for the benefits, the worker must find a job and work at least one period.)

The timing of events is as follows. At the very beginning of a period, a worker who was employed in the previous period must decide whether or not to quit. If she quits, she draws a new wage offer as described previously, and if she accepts the offer she immediately starts earning that wage without suffering any period of unemployment.

[10] **(c)** Write the Bellman equations for this problem. [Hint: Let $V_E(w)$ denote the value of a worker who was employed the previous period with wage w, before any decision to quit (and receive some new draw w') occurs.

$$V_E(w) = \max\left\{ w + \beta V_E(w), \int_{\underline{w}}^{\overline{w}} V_U^1(w')dF(w') \right\}$$

$$V_U^1(w) = \max\left\{ w + \beta V_E(w), \gamma + \beta \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w') \right\}$$

$$V_U^+(w) = \max\left\{ w + \beta V_E(w), \beta \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w') \right\}$$

[10] **(d)** Characterize (i.e. give an expression for it) the reservation strategy of an employed worker and then prove that the reservation wage for someone unemployed longer than one period is equal to 0 (if you cannot prove it, give some intuition as to why it must be the case).

First denote the reservation wage of an employed person as w_R^E (this gives the cutoff for when a worker will decide to quit or stay at a job). It is defined by the following.

$$w_R^E + \beta V_E(w_R^E) = \int_{\underline{w}}^{\overline{w}} V_U^1(w')dF(w')$$

Because the problem is the same every period for an employed worker, her quit / stay decision will always be the same. We can utilize this fact to simplify the above.

$$w_R^E = (1-\beta)\int_{\underline{w}}^{\overline{w}} V_U^1(w')dF(w')$$

Regarding the proof, suppose that $w_R^+ > 0$. Then we must have that

$$\underbrace{w_R^+ + \beta V_E^+(w_R^+) = \beta \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w')}_{\text{reservation statement for "+" unemployed}} < \underbrace{\int_{\underline{w}}^{\overline{w}} V_U^1(w')dF(w') = w_R^E + \beta V_E(w_R^E)}_{\text{reservation statement for employed workers}}.$$

Since $w+\beta V_E(w)$ is (weakly) increasing in w, the above statement necessarily implies that $w_R^+ < w_R^E$. Because we are talking about reservation strategies, if we were to plug in w_R^+ into the employed worker's problem, we know she will quit.

$$V_E(w_R^+) = \int_{\underline{w}}^{\overline{w}} V_U^1(w')dF(w')$$

Finally, we can utilize the above result to plug into the reservation statement for "+" workers (and do some moving around).

$$w_R^+ + \beta \overbrace{\int_{\underline{w}}^{\overline{w}} V_U^1(w')dF(w')}^{V_E(w_R^+)} = \beta \int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w')$$

$$w_R^+ = \beta \left[\int_{\underline{w}}^{\overline{w}} V_U^+(w')dF(w') - \int_{\underline{w}}^{\overline{w}} V_U^1(w')dF(w')\right] < 0 \qquad \text{(contradiction)}$$

The above is a contradiction, and so we cannot have $w_R^+ > 0$. Because wages are (weakly) positive, we must have $w_R^+ = 0$. Intuitively, since after the first period of unemployment a worker does not receive any benefits, accepting an offer and then quitting is at least as good as rejecting an offer and drawing again next period.