Instructor: *Javier Birchenall*
TA: *Charlie Nusbaum*
Econ 204C

Final

Problem 1. *Optimal risk sharing: homogeneous households.* [35pts.]

Consider a finite number I of ex-ante identical households that live infinite periods and receive a stochastic income that depends on the state of nature. In period t, the state of nature is $s_t \in S$, where S is a finite set. The history of states of nature up to period t is defined recursively by $s^t = (s_t, s^{t-1}) \in S^t$. Income for household i is $y_t^i(s_t)$. Aggregate income is $Y_t(s^t) \equiv \sum_{i=1}^I y_t^i(s_t)$.

Consumption in period t and history s^t is $c_t^i(s^t)$. Households maximize an expected utility function given by

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t U(c_t^i(s^t))\Pi(s^t),$$

where $\Pi(s^t)$ is the probability of history s^t and with a discount rate $0 < \beta < 1$.

Assume that a Social Planner solves the above allocational problem. The planner assigns a *Pareto weight* λ^i to household i, which we take as given. The purpose of the following questions is to examine the efficient sharing of risk under alternative storage arrangements, and specifications.

1) Assume that there are no savings or storage technologies. Write the Social Planner problem and show that the optimal consumption for household i in period t only depends on aggregate income during that period, $Y_t(s^t)$. That is, once we control for $Y_t(s^t)$ consumption is history-independent or independent of s^t. [HINT: For convenience, you may normalize the Lagrange multipliers at t and s^t by $\beta^t\Pi(s^t)$, i.e., use a multiplier $\tilde{\mu}(s^t) \equiv \beta^t\Pi(s^t)\mu(s^t)$].

The feasibility constraint is $Y_t(s^t) \equiv \sum_{i=1}^I y_t^i(s_t) \geq \sum_{i=1}^I c_t^i(s_t)$. This constraint is static so we can write the Social Planner problem as

$$\sum\nolimits_{i=1}^I \lambda^i \sum\nolimits_{t=0}^{\infty} \sum\nolimits_{s^t \in S^t} \beta^t U(c_t^i(s^t))\Pi(s^t) + \mu_t(s^t)\beta^t\Pi(s^t)\left[Y_t(s^t) - \sum\nolimits_{i=1}^I c_t^i(s_t)\right],$$

for all t and s^t. The first order condition for $c_t^i(s^t)$ is $\lambda^i U'(c_t^i(s^t)) = \mu_t(s^t)$ which defines $c_t^i(s^t) = [U']^{-1}(\mu_t(s^t)/\lambda^i)$. This term only depends on s^t through the multiplier. Using the feasibility constraint yields $\sum_{i=1}^I [U']^{-1}(\mu_t(s^t)/\lambda^i) = Y_t(s^t)$, which means that $c_t^i(s^t)$ is an implicit function of $Y_t(s^t)$ and $\{\lambda^i\}$, as in $c_t^i(s^t) = G(Y_t(s^t), \{\lambda^i\}_{i=1}^I)$.

2) Assume now that the Planner has access to a storage technology such that if $A_t(s^{t-1}) \geq 0$ units of consumption are set aside in period $t-1$, available resources expand by $(1+r_t(s^t))A_t(s^{t-1})$ in period t. These additional stored goods supplement current income. Let $C_t(s^t) \equiv \sum_{i=1}^I c_t^i(s_t)$ denote aggregate consumption. Modify the feasibility constraint accordingly and write the Social Planner problem. Show that a similar result as that derived in 1 holds here. In particular, show that consumption is history-independent in the sense that once we control for $C_t(s^t)$, **not** $Y_t(s^t)$, consumption is independent of s^t.

The feasibility constraint is now $(1+r_t(s^t))A_t(s^{t-1}) + Y_t(s^t) \geq A_{t+1}(s^t) + \sum_{i=1}^{I} c_t^i(s_t)$. This constraint is dynamic so we can write the Social Planner problem as

$$\sum_{i=1}^{I} \lambda^i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t U(c_t^i(s^t))\Pi(s^t) + \mu_t(s^t)\beta^t \Pi(s^t) \left[(1+r_t(s^t))A_t(s^{t-1}) + Y_t(s^t) \ldots \right.$$
$$\left. -A_{t+1}(s^t) - \sum_{i=1}^{I} c_t^i(s_t)\right],$$

for all t and s^t, which gives a first-order condition $\lambda^i U'(c_t^i(s^t)) = \mu_t(s^t)$ but with different values of $\mu_t(s^t)$ relative to part 1. That is, using the feasibility constraint yields $\sum_{i=1}^{I}[U']^{-1}(\mu_t(s^t)/\lambda^i) = C_t(s^t)$, which pins down the Lagrange multiplier so $c_t^i(s^t) = G(C_t(s^t), \{\lambda^i\}_{i=1}^{I})$, with endogenous values of $C_t(s^t)$.

Problem 2: *Optimal risk sharing: heterogeneous households.* [25pts]

So far we have assumed that households are identical. Assume now that households have different preferences summarized by

$$U^i(c_t^i(s^t)) = \frac{-1}{\gamma^i} \exp\{-\gamma^i c_t^i(s^t)\},$$

where γ^i differs across the population.

1) Assume that there is no storage technology (as in the first part of the previous question) and show that consumption satisfies

$$c_t^i(s^t) = a^i Y_t(s^t),$$

where a^i is a constant that satisfies $\sum_{i=1}^I a^i = 1$. What determines a^i?

The setting is the same as before so we have $\lambda^i U'(c_t^i(s^t)) = \lambda^i \exp\{-\gamma^i c_t^i(s^t)\} = \mu_t(s^t)$, or $c_t^i(s^t) = \ln[\mu_t(s^t)_t(s^t)/\lambda_i]/(-\gamma^i)$ so that $Y_t(s^t) = \sum_{i=1}^I \ln[\mu_t(s^t)/\lambda_i/\gamma^i]$, or

$$c_t^i(s^t) = \frac{1}{\gamma_i \sum \gamma_i}[\ln \lambda_i \sum \tfrac{1}{\gamma_i} - \sum \tfrac{\ln \lambda_i}{\gamma_i} + Y_t(s^t)]$$

Assume that $\lambda_i = \lambda_j \ \forall i, j$

$$c_t(s^t) = \frac{1}{\gamma_i \sum \gamma_i} Y_t(s^t)$$

2) Since γ^i orders tolerance to risk, show that more risk averse households (i.e., households with higher values of γ^i) should receive lower shares of aggregate income relative to less risk averse households.

Notice that a^i is decreasing in γ^i.

Problem 3: *Income-fluctuations problem.* [30pts.]

Consider an infinitely-lived household with a constant endowment of income $y > 0$, per period. The household's period utility $U(c)$ is well-behaved (i.e., strictly concave and increasing, and twice differentiable), and time is discounted at a rate $1/(1+\rho)$ with $\rho > 0$. Initial wealth is $a_0 > 0$ and there is a safe asset that yields $0 < r < \rho$ rate of return. Households can save but they cannot borrow, i.e., wealth cannot be negative.

1) Write the Bellman equation for the household problem. Characterize as much as possible the state space, the value function, and include any and all constraints. What properties does the Bellman equation satisfy? BRIEFLY (i.e. no proofs) explain why.

$$v(a) = \max_{0 \leq a' \leq a(1+r)+y} \{U((1+r)a + y - a') + \beta v(a')\}$$

2) Write the first-order condition and derive the Euler equation. Show that if wealth is carried over to the future period then consumption in the next period is lower than in the current period.

$[a'] : -U'((1+r)a + y - a') + \beta v'(a') \leq 0$ with equality iff $a' > 0$. $[a] : v'(a) = (1+r)U'((1+r)a + y - a')$ we have

$$U'(c) \geq \frac{1+r}{1+\rho} U'(c')$$

Since $a' > 0$ makes the previous equation hold with equality, we know that $U'(c) = \beta(1+r)U'(c')$ or $U'(c) < U'(c')$ so that $c > c'$.

3) Assume that the endowment of income is an AR(1) random variable. Write the Bellman equation for the household problem and informally describe a competitive equilibrium allocation in an exchange economy where assets are in zero net supply. A graphical description will suffice for your discussion of the competitive equilibrium.

$$v(a, y) = \max_{0 \leq a' \leq a(1+r)+y} \{U((1+r)a + y - a') + \beta \mathbf{E}_{y'}[v(a', y')|y]\}$$

and there is a precautionary demand for savings.